The reconstruction of planar graphs

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THE RECONSTRUCTION OF PLANAR GRAPHS

by

J. Lauri B.Sc., M.Sc.

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ABSTRACT

The object of this thesis is to investigate the Reconstruction Problem for planar graphs. This study naturally leads to related topics concerning certain nonplanar graphs and the use of their embeddings on appropriate surfaces to reconstruct them. The principal aim of this work is to find new techniques of reconstruction and to increase the number of classes of graphs known to be reconstructible. In achieving this aim, various important properties of graphs, such as connectivity and uniqueness of embeddings, are explored, and new results on these topics are obtained.

Part I, which consists of three chapters, contains a historical, non-technical introduction and general graph-theoretical definitions, notation and results. Some new concepts in reconstruction are also presented, notably the idea of reconstructor sets. Part II of the thesis deals with the vertex-reconstruction of maximal planar graphs: Chapter 4 is concerned with the vertex-recognition of maximal planarity, whereas Chapter 5 deals with the vertex-reconstruction. Part III deals with edge-reconstruction: planar graphs with minimum valency 5 and 4-connected planar graphs are reconstructed in Chapters 6 and 7 respectively. In Chapter 7, extensive use is made of the concept of reconstructor sets introduced in Chapter 3. This chapter also contains a brief discussion on the reconstruction of graphs from edge-contracted subgraphs, a problem which, in certain cases, can be regarded as dual to the Edge-reconstruction Problem.

Part IV is concerned with extending the results and techniques of the previous chapters to nonplanar graphs. Chapter 8 discusses where the previous techniques fail, and indicates where new methods are needed. In Chapter 9, all graphs which triangulate some surface and have connectivity 3 are edge-reconstructed. Certain graphs which triangulate the torus or the projective plane are also shown to be weakly vertex-reconstructible. Chapter 10 deals with the edge-reconstruction of all graphs which triangulate the projective plane.

The Appendix proves a conjecture of Harary on the cutvertex-reconstruction of trees. One technique used here ties up with a method employed in previous chapters on edge-reconstruction.

CONTENTS

PREFACE

This thesis presents an account of my research for the Ph.D. degree of the Open University. I should like to express my deepest gratitude to Dr. S. Fiorini, my supervisor, who introduced me to graph theory and who, during my course of study, was a constant source of help and encouragement. During my stay in England I was a Commonwealth Scholar under the Commonwealth Scholarship and Fellowship Plan. I should therefore like to express my gratitude to the Commonwealth Scholarship Commission in the United Kingdom and to the British Council. Without their financial support this research would not have been possible. Finally, I should also like to thank the Open University for their practical and financial support, and the staff at the South Western Regional Office for extending to me their kind hospitality.

Most of the material in this thesis has been submitted for publication. The results of Chapter 6 have appeared in:

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The results of Chapter 4 are to appear in:

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whereas those of Chapter 5 are to appear in:

J. Lauri, The reconstruction of maximal planar graphs, II: Reconstruction. *J. Combinatorial Theory (B)* ZBL*413*#05036

The main results of Chapter 7 have been submitted to the Journal of Graph Theory, whereas those of Chapters 9 and 10 have been submitted to the Quarterly Journal of Mathematics (Oxford), both as joint papers with S. Fiorini. Finally, the results of the Appendix on the cutvertex-reconstruction of trees have been submitted to Discrete Mathematics.

PART I PROLOGUE

In these introductory chapters we provide the necessary background for what follows. We start, in Chapter 1, by giving a brief historical, non-technical introduction to the Reconstruction Problem. In Chapter 2, we give basic graph-theoretical definitions and results which will be required later. This chapter includes a discussion of uniqueness of plane embeddings, a concept which will be of crucial importance in all that follows. In Chapter 3 we start the technical treatment of the Reconstruction Problem, giving the basic definitions and some standard results. In this chapter we also present some new concepts and results, notably the idea of reconstructor sets and reconstructor sequences which will be of prime importance when we study edge-reconstruction later on.

CHAPTER 1 *INTRODUCTION*

In this chapter we present a brief, non-technical introduction to the Reconstruction Problem. In view of a number of survey articles on the topic, notably the work by Bondy and Hemminger [BH1] and by Nash-Williams [BW1, Chapter 8], we shall not attempt to present a detailed catalogue of all that has been achieved in this area, but refer the reader to these two survey papers. What we shall attempt to do is to provide sufficient background information to make this thesis as self-contained as possible.

The Reconstruction Problem is regarded by many as one of the foremost unsolved problems in graph theory. It was discovered by Ulam who published [U1] a statement of the problem in 1960, although according to Harary [H3], it was already known to him in 1929. In terms of graphs, Ulam's problem is to determine whether or not the following conjecture is true:

> Suppose that G and H are simple graphs with vertex-sets $\{v_1,v_2,\ldots,v_\nu\}$ and $\{u_1,u_2,\ldots,u_\nu\}$ respectively, $\nu \geq 3$, and suppose that for each i, the subgraphs $G - v_i$ and $H - u_i$ are isomorphic. Then G and H are themselves isomorphic. †

The first attack on this problem appeared in [K2], where Kelly showed that the conjecture is true if G and H are trees. This result had been obtained in Kelly's doctoral thesis [K1] written under Ulam. Kelly also showed that the conjecture is true when the graphs are either regular or disconnected.

† Graph-theoretical terms used in this chapter will be explained in Chapter 2.

In [H1], Harary proposed an alternative way of posing the conjecture:

> Let G be a simple graph with vertex-set $\{v_1, v_2, \ldots, v_\nu\}$, $\nu \geq 3$, and let the family (called the vertex-deck) of the subgraphs $G - v_i$ be given. Then the graph G can be reconstructed uniquely, up to isomorphism, from these vertex-deleted subgraphs.

This formulation of the conjecture led to Ulam's problem being referred to as the Reconstruction Problem.

Apart from this original form of the Reconstruction Problem, various other problems have been posed, dealing with the reconstruction of graphs from information other than the vertex-deleted subgraphs. For example, one might ask whether or not a graph is uniquely reconstructible from its edge-contracted subgraphs, or from its subgraphs obtained by identifying pairs of non-adjacent vertices. However, the most natural of these variations is the problem which asks whether or not the following conjecture is true:

> Let G be a simple graph with edge-set $\{e_1, e_2, \ldots, e_\varepsilon\}$, $\varepsilon \geq 4$, and let the family (called the edge-deck) of the subgraphs $G - e_i$ be given. Then the graph G can be reconstructed uniquely, up to isomorphism, from these edge-deleted subgraphs.

This edge form of the Reconstruction Problem, posed by Harary in [H1], has received as much attention as the original form of the problem. It is with these two forms of the Reconstruction Problem that we shall be primarily concerned in this thesis.

Apart from Kelly's work, the Reconstruction Problem did not attract much attention before the mid-late 1960s, but since then, the literature on the problem has increased at a rapid rate (see [BH1]).

Results by Harary and Palmer in 1965 and by Greenwell in 1971 confirmed the intuitive feeling that the vertex version of the Reconstruction Problem is stronger than the edge version, by showing that if the first conjecture is true then so is the second. In fact, Greenwell's result states that if a graph with no isolated vertices is vertex-reconstructible (that is, reconstructible from its vertex-deleted subgraphs), then it also edge-reconstructible. Thus, by combining this with results on vertex-reconstruction (like Kelly's result on trees), one immediately obtains edge-reconstructible classes of graphs.

Some of the earlier work on the problem was concerned with improving Kelly's result on trees by showing that not all the vertex-deleted subgraphs of a tree are required to reconstruct it uniquely. On these same lines Harary has recently made a further conjecture which we prove in the Appendix.

However, most of the work on reconstruction deals of course with graphs other than trees. Instead of trying to solve the problem at one fell swoop, most researchers, following Kelly's footsteps, devote their attention either to reconstruct parameters of graphs or to reconstruct classes of graphs. By reconstructing parameters (such as connectivity and the valency list, say) one is retrieving from the vertex-deck or the edge-deck valuable information about the original graph, which could ultimately lead to reconstruction. When reconstructing classes of graphs, one hopes that eventually enough classes will be found to include all graphs. The classes of graphs which are known to be vertex-reconstructible (and hence edge-reconstructible) are not many, and most of them are simple in structure, with low connectivity and with tree-like properties (see [BH1] for more details). Hence it seems desireable to attempt the reconstruction of other, less simple, classes of graphs; one such class is that of planar graphs. Furthermore, since any graph is embeddable on some surface, the study of the reconstruction of planar graphs can be regarded as a possible first step towards a systematic reconstruction of all graphs by exploiting their embeddings on surfaces.

Although some graph theorists believe that the vertex form of the Reconstruction Conjecture might be false, it seems that there is general consensus as to the truth of the edge form of the conjecture. Since this latter conjecture is a weaker version of the former, one would expect that there are edge-reconstruction results not available in vertex-reconstruction. The most striking of such results is Lovasz's theorem (1972), improved by Müller in 1977, which states that a graph with ν vertices is edge-reconstructible if it has more than $(\nu\log\nu)/(\log 2)$ edges. Since the maximum possible number of edges of a graph on ν vertices is $\frac{1}{2}\nu(\nu-1)$, one might say that, for large ν, almost all graphs on ν vertices are edge-reconstructible. The most notable feature of this result is its elegant and short proof which uses a very ingenious application of the inclusion-exclusion principle.

Analogues of these two Reconstruction Conjectures have been posed and studied for other structures apart from simple, undirected, finite graphs. Perhaps the most interesting of these, and the one which might have most bearing on the reconstruction of graphs is the problem of reconstructing digraphs. While Müller's result on the one hand generally provides ammunition for those who believe in the truth of the Reconstruction Conjectures, on the other hand one finds Stockmeyer's disconcerting discovery that the analogue of the vertex form of the Reconstruction Conjecture is false for digraphs. In fact, non-reconstructible tournaments on five and six vertices had been found by Beineke and Parker in 1970, but the final stroke was administered by Stockmeyer in 1976 and 1977 when he exhibited an infinite class of non-reconstructible tournaments.

In view of these counterexamples and similar negative conclusions for other structures like matroids and infinite graphs (see [BH1]), in the words of Schwenk [S1] , "No longer does it appear that we .... are nibbling away at a grand yet almost certain truth. Instead, we can now recognize that reconstructibility is necessarily limited, and we can proceed with our newly obtained perspective to try to map out those limits."

CHAPTER 2 BASIC DEFINITIONS AND RESULTS

In this chapter we shall give those basic definitions and results which will be used in this thesis and which are of a general graph-theoretic nature; other definitions pertaining more directly to the Reconstruction Problem will be given in the next chapter. More specialized terms whose definitions are not included in these two chapters will be defined as they appear.

SECTION 2.1 - GRAPHS AND SUBGRAPHS

A graph G is a pair (VG,EG) where VG is a finite set of vertices and EG is a finite family of unordered pairs of (not necessarily distinct) vertices; the elements of EG are called edges. The order of G, denoted by νG is the number of vertices of G; the number of edges is denoted by εG.

If $e = \{u,v\}$ is an edge of G, then e is said to join the vertices u and v, and each one of u and v is said to be incident to e, and u and v are said to be adjacent. If two edges of G are incident to a common vertex, then they are also said to be adjacent. For convenience, the edge joining u and v will be denoted by uv or vu.

Two or more edges of G joining the same pair of vertices are called multiple edges, and an edge vv is said to be a loop. A graph with no loops or multiple edges is called a simple graph. To emphasise that a particular graph under discussion might have loops or multiple edges we sometimes refer to it as a general graph. The complement G^c of a simple graph G is the graph with the same vertex-set as G, but where two vertices are adjacent if and only if they are not adjacent in G. We emphasize here that throughout this thesis, any graph considered will be simple unless otherwise stated. Many of the concepts defined below for simple graphs have an obvious extension to general graphs.

For each vertex v of G, the number of edges of G incident to v is called the <u>valency</u> of v in G, denoted by $\rho_G v$, or simply by ρv, if it is clear from the context that we are referring to valencies in G. A vertex of valency k is called a <u>k-vertex</u>. The number of k-vertices of G is denoted by $\nu_k G$ or ν_k. For $v \in VG$ we denote by $N_G v$ the set of <u>neighbours</u> of v in G, that is, the set of those vertices adjacent to v in G. Here and in similar cases we usually drop the reference to G when the context is clear. The maximum valency in G is denoted by ΔG or Δ; the minimum valency is denoted by δG or δ. The family $\{\{\rho_G v: v \in VG\}\}$ is called the <u>valency list</u> of G. The <u>neighbourhood valency list</u> of a vertex v in G is the family $\{\{\rho_G w: w \in N_G v\}\}$. †

A <u>subgraph</u> of a graph $G = (VG,EG)$ is a graph $H = (VH,EH)$ such that $VH \subseteq VG$ and $EH \subseteq EG$; we sometimes denote this by writing $H \subseteq G$. If $Q \subseteq VG$, then the subgraph $\langle Q \rangle$ <u>induced by Q</u> is that subgraph H of G for which $VH = Q$, and $EH = \{vw \in EG: v,w \in Q\}$. If $W \subseteq EG$, then the subgraph $\langle W \rangle$ <u>induced by W</u> is that subgraph K of G for which $VK = \{v \in VG: vx \in W$ for some $x \in VG\}$ and $EK = W$. The subgraph $\langle VG - Q \rangle$ is denoted by $G - Q$; $\langle EG - W \rangle$ is denoted by $G - W$. In obtaining $G - Q$ from G we say that the vertices of Q have been <u>deleted</u> from G; similarly for $G - W$ we say that the edges of W have been deleted from G. If $Q = \{v\}$ and $W = \{e\}$

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† The usual symbol $\{\cdot\}$ will denote a set, whereas $\{\{\cdot\}\}$ will denote a family, where it is understood that two elements of a family need not be distinct.

we often denote the vertex-deleted subgraph $G - v$ and the edge-deleted subgraph $G - e$ by G_v and G_e respectively. If $v,w \in VG$ and $vw \notin EG$, we then denote by $G + vw$ that graph obtained from G by joining v and w by an edge.

Let H be a subgraph of G. A vertex of contact of H in G is a vertex of H that is adjacent in G to a vertex not belonging to VH. The set of vertices of contact of H in G is denoted by $C(G,H)$.

If $vw \in EG$, then vw is said to be subdivided if vw is deleted and replaced by edges vx and xw, where x is a new vertex. A subdivision of G is a graph that can be obtained from G by a possibly empty sequence of subdivisions of edges. The edge vw is said to be contracted if it is deleted and the vertices v and w identified; the resulting graph is denoted by $G.vw$. Note that although G is a simple graph, an edge-contracted subgraph $G.e$ of G need not be simple.

If G and H are two graphs, then we say that G is contractible to H if H can be obtained from G by a possibly empty sequence of edge-contractions.

Let $W = \{v_i v_{i+1}: i = 0,1,\ldots,t-1\}$ be a set of edges of G, where all the vertices v_i, except possibly v_0 and v_t, are distinct. If $v_0 \neq v_t$, then we shall call the subgraph $\langle W \rangle$ of G a chain from v_0 to v_t, and we shall denote it by $C = C[v_0,v_t]$. We shall also say that v_0 and v_t are joined by the chain C. The vertices $v_1, v_2, \ldots, v_{t-1}$ are called the internal vertices of the chain. The length of the chain is t. An edge is therefore a chain of length 1. The C-distance between v_i and v_j, $0 \le i \le j \le t$ is $(j - i)$. We denote $C[v_0,v_t] - v_0$, $C[v_0,v_t] - v_t$ and $C[v_0,v_t] - v_0 - v_t$ by $C]v_0,v_t]$, $C[v_0,v_t[$ and $C]v_0,v_t[$ respectively. A set of chains in G is said to be internally disjoint if no vertex of G is an internal vertex of more than one chain of the set. If $v_0 = v_t$, then $\langle W\rangle$ is said to be a circuit, or a t-circuit if we want to indicate the number of vertices in it. A triangle is a 3-circuit. If $\langle W\rangle$ is a circuit such that $v,w \in V\langle W\rangle$ and $vw \in EG - E\langle W\rangle$ we then say that the edge vw is a chord of $\langle W\rangle$. We often denote $\langle W\rangle$ by $v_0v_1v_2\dots v_t$ (whether or not v_0 and v_t are distinct).

A graph on n vertices is called Hamiltonian if it contains an n-circuit. The complete graph, that is the simple graph with n vertices and $\frac{1}{2}n(n-1)$ edges is denoted by K_n. The complete graph with an edge deleted is denoted by $K_n - e$. A bipartite graph is one whose vertex-set can be partitioned into two sets in such a way that each edge joins a vertex of the first set to a vertex of the second. A complete bipartite graph is a bipartite graph in which every vertex in the first set is joined to every vertex in the second set; if the two sets contain r and s vertices respectively, then the complete bipartite graph is denoted by $K_{r,s}$.

A forest is a graph in which every pair of vertices is joined by at most one chain. A tree is a forest in which every pair of vertices is joined by at least one chain.

Two general graphs G and H are said to be isomorphic if there is a bijection $\psi: VG \to VH$ such that the number of edges joining v and w in G is equal to the number of edges joining ψv and ψw in H. The function ψ is said to be an isomorphism from G to H. When G and H are isomorphic we denote this by $G \simeq H$. An automorphism of G is an isomorphism $\psi: VG \to VG$.

To every isomorphism $\psi: VG \to VH$ there corresponds an edge-isomorphism $\psi': EG \to EH$ such that $\psi'(uv) = \psi u \psi v$, giving that e and f are adjacent edges in G if and only if $\psi' e$ and $\psi' f$ are adjacent in H.

If K is a subgraph of G, then by ψK we mean that subgraph of H for which $V(\psi K) = \psi(VK)$ and $E(\psi K) = \psi'(EK)$.

(So far, we have consistently avoided unnecessary use of brackets, preferring to write, for example, VG and δG instead of $V(G)$ and $\delta(G)$ respectively. We shall always do this unless there is a special need for brackets, as in $V(\psi K)$ or $\delta(G-v)$, say.)

SECTION 2.2 - CONNECTIVITY

A graph is connected if every pair of vertices is joined by at least one chain. A maximal connected subgraph of G is called a component of G. A cutvertex of G is a vertex whose deletion increases the number of components. If G is not connected, we then say that it is disconnected.

A connected graph is said to have connectivity $\kappa = \kappa G$ if the deletion of some set of κ vertices disconnects G, and κ is the least integer with this property. If G is K_n, then κG is by definition taken to be $n-1$; when $\kappa G = 1$ we say that G is separable. For any $k \leq \kappa$, G is said to be k-connected. Any set Q of vertices of G whose deletion disconnects G is said to be a separating set of G. The number of separating sets of G having r vertices is denoted by $s_r G$. Also, the set Q is said to separate the vertices u and v of G if u and v are in different components of $G - Q$, or equivalently if any chain from u to v in G contains at least one vertex of Q. If C is a circuit in G and VC is a separating set of G we often say that C is a separating circuit. Let Q be a separating set of G and let the components of $G - Q$ be $H_1, H_2, \ldots, H_r$. Then, for $1 \leq i \leq r$, the subgraph of G induced by $VH_i \cup Q$ is denoted by $\overline{H}_i$.

We shall need the following fundamental result on connectivity due to Menger [M3].

<u>Theorem 2.1</u> (Menger)

If u and v are distinct nonadjacent vertices of a graph G, then the maximum number of internally disjoint chains from u to v in G equals the minimum number of vertices of G that separate u and v. □

We shall also need a second version of Menger's Theorem, which gives a characterization of k-connected graphs. This theorem, which follows from Theorem 2.1, was proved independently by Whitney [W3].

<u>Theorem 2.2</u> (Whitney-Menger)

A graph G is k-connected if and only if every pair of distinct vertices are joined in G by at least k internally disjoint chains. □

One last result on connectivity which we shall find useful is the following theorem, proved in [CKL1].

<u>Theorem 2.3</u> (Chartrand-Kaugars-Lick)

If G is a k-connected graph whose minimum valency δ satisfies $\delta \geq \frac{1}{2}(3k - 1)$, then there exists a vertex v of G such that G_v is also k-connected. □

SECTION 2.3 - PLANARITY

In this section we shall be using some standard topological terminology. For explanation of undefined terms the reader is referred to [AS1] and [BW1, Chapter 2].

<u>Embeddings of graphs in surfaces</u>

By a <u>closed surface</u> we shall mean a connected, compact topological space S such that for every point x of S, x has a neighbourhood in S homeomorphic to an open disk (or "2-cell"). All surfaces we consider will be closed surfaces. To conform with our definition of a surface, by the plane we shall mean the extended plane (that is $\mathbb{R}^2$ with the point ∞ adjoined to it) with the usual compactification (see[AS1, p.3]). Hence the plane is homeomorphic to the sphere.

Let G be a graph with $VG = \{v_1, v_2, \ldots, v_\nu\}$ and $EG = \{e_1, e_2, \ldots, e_\varepsilon\}$. An embedding or representation of G in a surface S is a subspace G_S of S such that $G_S = \{v_1(S), \ldots, v_\nu(S)\} \cup \{e_1(S), \ldots, e_\varepsilon(S)\}$, where

(i) $v_1(S), \ldots, v_\nu(S)$ are distinct points of S,

(ii) $e_1(S), \ldots, e_\varepsilon(S)$ are mutually disjoint open arcs in S,

(iii) $v_i(S) \cap e_j(S) = \emptyset$, $i=1,2,\ldots,\nu$; $j=1,2,\ldots,\varepsilon$,

(iv) if $e_j = v_{j1}v_{j2}$, then the open arc $e_j(S)$ has $v_{j1}(S)$ and $v_{j2}(S)$ as endpoints, $j=1,2,\ldots,\varepsilon$.

(Here an open arc in S means a homeomorphic image of the open interval $]0,1[$.)

It is well-known that every surface S permits a triangulation K_g (see [AS1]) that is, S is homeomorphic to a geometric 2-dimensional simplicial complex K_g which satisfies Theorem 22E of Chapter I of [AS1]. (In fact, since we define our surfaces to be compact, the number of simplexes of K_g is finite.) We can therefore give the following alternative (equivalent) definition of embeddings. An embedding of a graph G in S is a subgraph L of the 1-skeleton K_g^1 of a triangulation K_g of S, such that G is isomorphic to L. If $L = K_g^1$, we then say that G triangulates S. If G triangulates S, then any embedding of G in S is the 1-skeleton of some triangulation of S. When G triangulates the plane we say that G is maximal planar. In general, if G is embeddable in the plane we say that G is planar; whereas if G is embeddable in the projective plane P we say that G is projective. If G is nonplanar but G_v is planar for every vertex v of G, then we say that G is critical.

REMARK. For convenience and simplicity of notation, we shall often write G for its topological realization G_S (or L) on S, and we shall designate the point $v_i(S)$ and the open arc $e_j(S)$ by v_i and e_j respectively, for $i=1,2,\ldots,\nu$ and $j=1,2,\ldots,\varepsilon$, referring to them as vertices and edges of G.

The connected regions of $S - G$ are called <u>faces</u> of the embedding of G on S. If F is a face and the vertex v is in the <u>boundary</u> $(\overline{F} - F)$ of F, then we say that v is <u>incident</u> to F. A similar definition holds for an edge incident to F. If F is homeomorphic to the open disk, and the boundary of F is a k-circuit, then we say that F is a <u>k-face</u>. In this case we also say that the <u>face-valency</u> of F is k, and we denote it by ρ^*F. A 3-face is sometimes called a <u>triangular face</u>. The k-circuit which is the boundary of F is said to be the <u>boundary circuit</u> of F and is also said to <u>bound</u> F. The <u>face-valency list</u> of the embedding is the family of the face-valencies of all the faces of the embedding. The maximum and minimum face-valencies of an embedding G_S are denoted by Δ^*G_S and δ^*G_S respectively.

If all the faces of an embedding of G in S are homeomorphic to an open disk, then the embedding is said to be a <u>2-cell embedding</u>. If a graph G with ν vertices and ε edges has a 2-cell embedding in a surface S with Euler characteristic χ, and if ϕ is the number of faces of the embedding, then Euler's formula states that

$$\nu + \phi = \varepsilon + \chi.$$

It follows that $\varepsilon \le 3\nu + 3\chi$, with equality holding if and only if G triangulates S. This inequality can also be written as

$$\sum_{k=1}^{5} (6 - k)\nu_k \;\ge\; 6\chi \;+\; \sum_{k=7}^{\Delta G} (k - 6)\nu_k,$$

and will be called Euler's inequality. (We note here that if G

triangulates S, then any embedding of G in S is a 2-cell embedding (see [BW1, Theorem 6.1 of Chapter 2] or [Y1]).)

In this thesis we shall be repeatedly referring to the following crucial fact which follows from (E2) of Theorem 22E in [AS1, Chapter I]:

> If G triangulates some surface, then for every vertex $v \in VG$, the subgraph $\langle Nv \rangle$ of G is Hamiltonian.

From this we deduce the following results.

Lemma 2.1

Let G be a graph which triangulates some surface and has connectivity κ. Let Q be a set of κ vertices of G whose deletion disconnects G. Then each vertex of Q has valency at least 2 in $\langle Q \rangle$.

Proof

Let $G_1, G_2, \ldots, G_r$, $r \geq 2$, be the components of $G - Q$, and let $v \in Q$. Since the connectivity of G is κ (and so $Q - \{v\}$ cannot be a separating set of G), then v has neighbours in each one of $G_1, \ldots, G_r$. Let $C = v_0 v_1 \ldots v_{\rho v-1} v_0$ be a Hamiltonian circuit of Nv. We may assume with no loss of generality that $v_0 \in VG_1$. Let v_p be the first vertex in the sequence $v_0, v_1, \ldots, v_{\rho v-1}$ which is not in VG_1, and let v_q be the last such vertex. We note that v_p and v_q are distinct. Otherwise all the vertices of $Nv - \{v_p\}$ would be in G_1, and so, since v must have neighbours in each one of $G_1, \ldots, G_r$, it would follow that $r = 2$ and $v_p \in VG_2$. But then, the vertex $v_{p-1} \in VG_1$ would be adjacent to the vertex $v_p \in VG_2$, contradicting the fact that G_1 and G_2 are distinct components of $G - Q$.

Now, the vertices v_{p-1} and v_{q+1} (not necessarily distinct) are in VG_1, and there exist edges $v_{p-1}v_p$ and $v_q v_{q+1}$ in C with $v_p, v_q \notin VG_1$. Hence, since there can be no edge joining a vertex of G_1 to a vertex of G_i, $i \neq 1$, we deduce that v_p and v_q are in Q. Therefore v has at least two neighbours in Q, as required. □

The following two corollaries are immediate consequences of Lemma 2.1

Corollary 2.1

If the graph G triangulates a surface, then G is 3-connected. □

Corollary 2.2

If the graph G triangulates a surface and Q is a separating 3-set of vertices of G, then $\langle Q \rangle$ is K_3. □

Embeddings in the plane

In this thesis we shall be primarily concerned with planar graphs. Here we shall give an account of the major results and concepts on planarity referred to in our work.

One of the most important of these results is Kuratowski's Theorem [K3] giving a characterization of planar graphs.

Theorem 2.4 (Kuratowski)

A graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$. □

The following analogous characterization, due to Wagner [W1] and Harary and Tutte [HT1] will also be required.

Theorem 2.5 (Wagner, Harary-Tutte)

A graph is planar if and only if it contains no subgraph contractible to K_5 or $K_{3,3}$. □

In Chapter 4 we shall be encountering outerplanar graphs. A graph is outerplanar if it can be embedded in the plane in such a way that all the vertices are incident to a common face. The following characterization of outerplanar graphs is found in [CH1].

Theorem 2.6 (Chartrand-Harary)

A graph is outerplanar if and only if it contains no subdivision of K_4 or $K_{2,3}$. □

An embedding of a planar graph G in the plane is called a plane representation or plane embedding of G. Such a representation is referred to as a plane graph. If C is a circuit of G, then in any plane representation of G, the circuit C partitions the plane into two open regions, the interior of C, denoted by IntC, and the exterior of C, denoted by ExtC, where ExtC is defined to be the unbounded region. A plane embedding of a 2-connected planar graph is called a k-representation ($k \geq 4$) if all the faces of the embedding, except one, are 3-faces, the exceptional face being a k-face. (Such graphs are also considered by Tutte [T2] in the context of enumeration of planar graphs. He refers to them as "near-triangulations".) A planar graph which has such a plane embedding is called k-representable.

Uniqueness of plane representations

In most of our work we shall be reconstructing certain planar graphs by making use of their plane representations. Since our aim will be to show uniqueness of reconstruction, it would be helpful if we could identify properties of the graphs which are independent of their plane representations. Indeed, our task will be that much simpler if we could show that the graphs have representations which are unique in a sense which we shall now make precise.

In [W3], Whitney proved the very important result that a 3-connected planar graph has a unique plane representation in the sense that G has a unique dual, that is, if R_1 and R_2 are any two plane representations of G, then the duals R_1^* and R_2^* are isomorphic. (For a definition of dual see [BW1]. We note that in general the dual of a simple plane graph need not be simple.) We shall use a different definition of uniqueness of embeddings. This definition follows closely that given in [O1].

Definitions

Two plane representations R_1 and R_2 of a 2-connected planar graph G are said to be plane equivalent, or simply referred to as equivalent, if there exists an automorphism ψ on G such that C is the boundary circuit of a face in R_1 if and only if ψC is the boundary circuit of a face in R_2. If all plane representations of G are equivalent, we then say that G has a unique plane representation.

REMARK. In this definition, the restriction to 2-connected planar graphs is made solely for convenience (see also last paragraph of p.16 in [O1]), and at any rate, we shall only be concerned with the uniqueness or otherwise of plane representations of 2-connected graphs. In the definition, if G is not assumed to be 2-connected, then the phrase "boundary circuit of a face" has to be replaced by the phrase "boundary of a face".

We note here that, although it might not be immediately evident from [O1], Ore's definition of equivalent plane representations is slightly different from ours in that for G to have a unique plane representation Ore requires that for *any* automorphism ψ of G and for *any* two plane representations R_1 and R_2 of G, C is a boundary circuit of a face in R_1 if and only if ψC is a boundary circuit of a face in R_2. (This difference explains the "necessary" part of Theorem 2.4.2 of [O1].) We shall call such an automorphism a face-preserving automorphism. Thus, the graph for which two plane representations are shown in Figure 2.1 has a unique plane representation by our definition, but not by Ore's, since no automorphism of the graph is face-preserving. (For example, the circuit abcda which bounds a face in R_1 is mapped by the identity automorphism into a circuit which does not bound a face in R_2.)

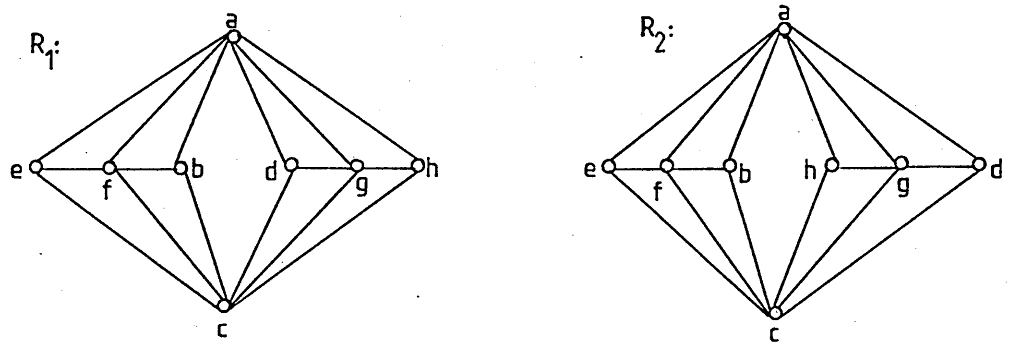


Figure 2.1

We said above that a 3-connected planar graph has a unique dual. It is also true that such a graph has a unique plane representation.

Theorem 2.7

If G is a 3-connected planar graph then any automorphism of G is a face-preserving automorphism, and so G has a unique plane representation.

In fact, this result is the "sufficient" part of Theorem 2.4.2 of [01]. We shall reproduce the proof below, after we have given some preliminary definitions which will be needed later in the thesis. However, before proceeding we make the following observations.

If R_1 and R_2 are two equivalent plane representations of a planar graph G it is not difficult to see that the duals R_1^* and R_2^* are isomorphic. However, when G is not 3-connected, then the fact that R_1^* and R_2^* are isomorphic does not necessarily imply that R_1 and R_2 are equivalent. This is illustrated in Figure 2.2. Although R_1^* is isomorphic to R_2^* here, R_1 is not equivalent to R_2, as we now show. By considering valencies of a and b and of their neighbours, we see that any automorphism must map b into b, and a into a.

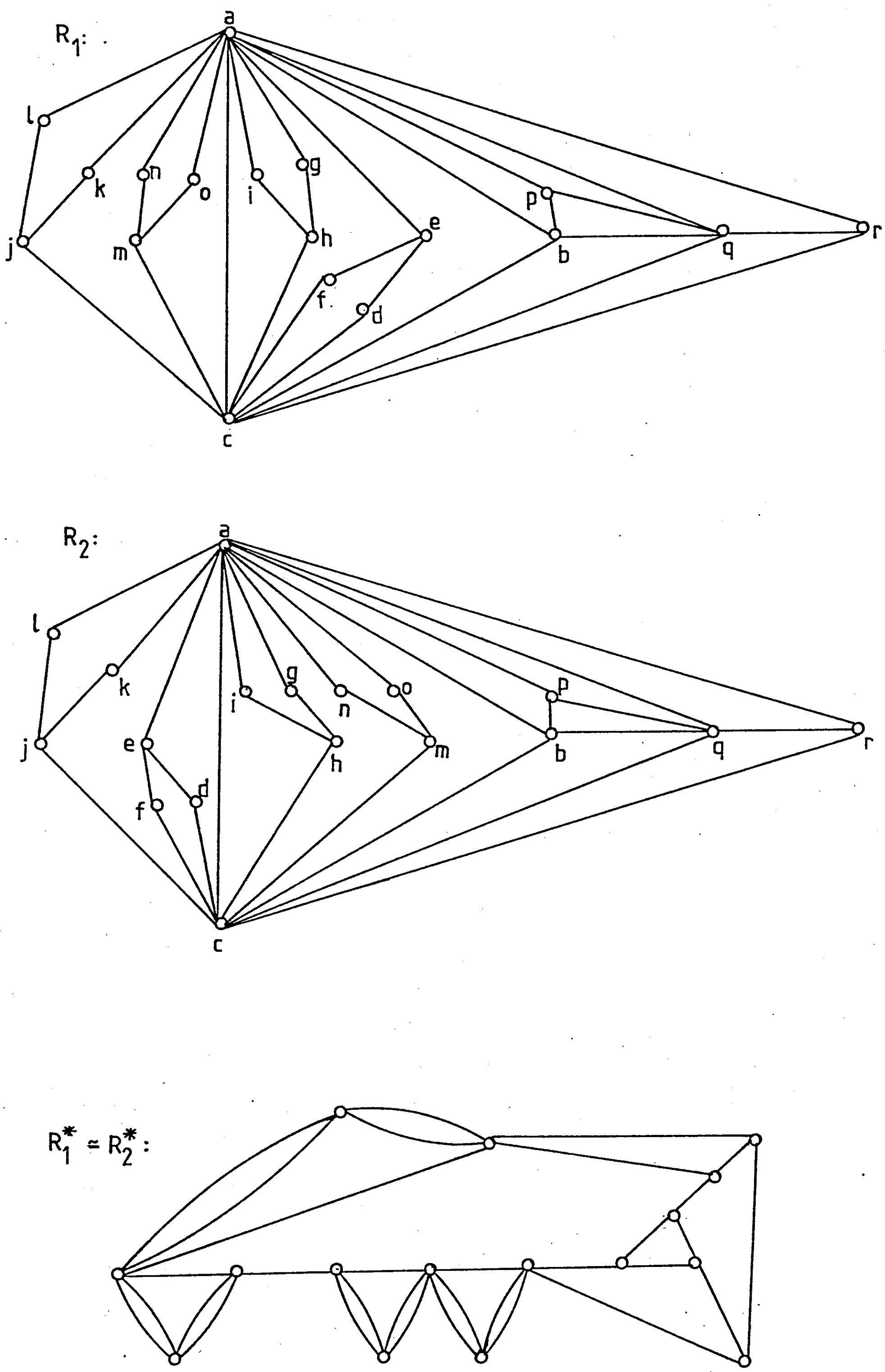


Figure 2.2

Therefore if R_1 and R_2 were equivalent there would be an isomorphism ψ such that if C is the circuit abcdea, which bounds a face in R_1, then ψC must be a circuit which bounds a face in R_2. Hence ψC can only be abcmoa, with $\psi a = a$ and $\psi b = b$, which is clearly impossible.

In view of Theorem 2.7, this situation obviously cannot arise when G is 3-connected. Indeed, one can prove idependently of Theorem 2.7 that if R_1 and R_2 are two plane representations of a 3-connected graph such that $R_1^* \simeq R_2^*$, then R_1 is equivalent to R_2 (see Lemma 1 of [T1]). This, together with Whitney's result that a 3-connected planar graph has a unique dual, gives a proof, different from that in [O1], that a 3-connected planar graph has a unique plane representation.

Bridges

When studying planar graphs, certain subgraphs, called bridges, play an important rôle. In fact we shall be making extensive use of bridges in our work. The theory of bridges in planar graphs is dealt with in detail in [BM1, Section 9.4] and especially in [O1, Chapter 2]. We shall here limit ourselves to some of the more basic definitions.

We shall only be interested in bridges of circuits in planar graphs. Thus, if C is a circuit of a planar graph G, then a <u>C-avoiding chain</u> is a chain in which no edges or internal vertices belong to C. Two edges e_1 and e_2 are said to be <u>connected outside of C</u> if there is a C-avoiding chain whose terminal edges are e_1 and e_2 respectively. Under these conditions e_1 and e_2 are said to be <u>bridge-equivalent</u> with respect to C. It is easy to see that bridge-equivalence is an equivalence relation on $EG - EC$; the equivalence class consisting of all edges that are bridge equivalent to an edge e with respect to C is said to form a <u>bridge</u> B for C.

A bridge can only have vertices in common with C. Such vertices will be called vertices of attachment of B with C. Any two vertices of B are joined by a chain having internal vertices in $VB - VC$ only. If B and B' are two bridges of a circuit C, u and v are two vertices of attachment of B with C, and u', v' two vertices of attachment of B' with C, and if moreover the four vertices are distinct and appear in the cyclic order u, u', v, v' on C, we then say that B and B' are skew. If G is a plane graph and C is a circuit of G such that the edges of the bridge B of C are in IntC, we then say that B is an inner bridge of C, whereas if the edges of B are in ExtC, we then say that B is an outer bridge of C.

We can now give the proof of Theorem 2.7.

Proof of Theorem 2.7

Let us assume that the theorem is not true and that ψ is an automorphism of G which is not face-preserving. Then there exists a circuit C, bounding a face in some plane representation R of G, such that in some other representation R', ψC is not the boundary circuit of any face. But then ψC has both inner and outer bridges in R', so that ψC, and hence C, has more than one bridge. Thus R has a face whose boundary circuit has more than one bridge. Therefore G is not 3-connected (see Theorem 2.4.1 of [01]). This contradiction proves the theorem. □

In view of this theorem, when G is a 3-connected planar graph we may assume with no loss of generality that G is a plane graph. In particular, since any plane embedding of G has the same face-valency list, it makes sense to talk about the face-valency list of G.

Finally we make some further remarks on standard notation. If A is a finite set, we denote the number of elements of A by $|A|$; if $|A| = k$, we say that A is a <u>k-set</u>. If A and B are sets, then A - B denotes that set containing all the elements of A which are not in B. The symbol := indicates that the equation in which it occurs acts as the definition of the expression on the left-hand side of the equation. The group of integers under addition modulo r is denoted by Z_r. The end (or absence) of a proof is denoted by the symbol $\square$.

CHAPTER 3 *THE RECONSTRUCTION PROBLEM*

In this chapter we shall give the basic results and definitions of reconstruction theory which will be used in this thesis. We shall also present some new concepts and results. Any result given here without proof is found either in [BH1] or [BW1, Chapter 8].

We shall be primarily concerned with two forms of the Reconstruction Problem, the Vertex-reconstruction Problem and the Edge-reconstruction Problem.

The vertex-deck † of a graph G is the family $\mathcal{D}G := \{\{G_v : v \in VG\}\}$. A graph H is a vertex-reconstruction of G if $\mathcal{D}G = \mathcal{D}H$. The graph G is said to be vertex-reconstructible if every vertex-reconstruction of G is isomorphic to G, that is, if the vertex-deck $\mathcal{D}G$ determines G uniquely,up to isomorphism.

The Vertex-reconstruction Conjecture

All graphs with at least three vertices are vertex-reconstructible.

The Vertex-reconstruction Problem is to determine the truth or falsity of the Vertex-reconstruction Conjecture.

The edge-deck of a graph G is the family $\mathcal{D}'G := \{\{G_e : e \in EG\}\}$. A graph H is an edge-reconstruction of G if $\mathcal{D}'G = \mathcal{D}'H$. The graph G is said to be edge-reconstructible if every edge-reconstruction of G is isomorphic to G, that is, if the edge-deck $\mathcal{D}'G$ determines G uniquely, up to isomorphism.

The Edge-reconstruction Conjecture

All graphs with at least four edges are edge-reconstructible.

† The term "deck" was first used by Harary in [H1].

The Edge-reconstruction Problem is to determine whether or not the Edge-reconstruction Conjecture is true.

Henceforth, whenever we consider the Vertex-reconstruction (Edge-reconstruction) Problem we shall always confine our attention to graphs with at least three vertices (four edges).

Short of trying to prove the conjectures directly, most attempts on the Reconstruction Problem fall into one of two categories. One is the reconstruction of parameters. A parameter is <u>vertex-reconstructible</u> (<u>edge-reconstructible</u>) for a class $\mathcal{J}$ of graphs if for each graph G in $\mathcal{J}$, it takes the same value on all vertex-reconstructions (edge-reconstructions) of G, that is, if the parameter can be determined from $\mathcal{D}G$ (or $\mathcal{D}'G$). The other category is the reconstruction of classes of graphs. A class $\mathcal{J}$ of graphs is <u>vertex-reconstructible</u> (<u>edge-reconstructible</u>) if each graph in $\mathcal{J}$ is vertex-reconstructible (edge-reconstructible).

Parameters which are vertex-reconstructible and edge-reconstructible include the order, the number of edges and the valency list. Moreover, given any G_v in $\mathcal{D}G$, one can determine $\rho_G v$ and also the neighbourhood valency list of v in G (see [M1] or [BW1, Chapter 8]). Similarly, $\{\{\rho_G w$: w incident to e in $G\}\}$ can be determined for each G_e in $\mathcal{D}'G$.

The following theorem, known as Kelly's Lemma, is a fundamental result in reconstruction theory, and turns out to be very useful in our work.

<u>Theorem 3.1</u> (Kelly's Lemma)

For any two graphs F and G such that $\nu F < \nu G$, the number $s(F,G)$ of subgraphs of G isomorphic to F is reconstructible from $\mathcal{D}G$, and moreover, given any $G_v \in \mathcal{D}G$, the number of subgraphs of G isomorphic to F and containing the vertex v is also reconstructible from G. Similarly, if $\varepsilon F < \varepsilon G$, then $s(F,G)$ is reconstructible from $\mathcal{D}'G$, and moreover, given any $G_e \in \mathcal{D}'G$, the number of subgraphs of G isomorphic to F and containing the edge e is also reconstructible from $\mathcal{D}'G$. □

The next theorem, due to Greenwell, gives a relationship between the Vertex-reconstruction Problem and the Edge-reconstruction Problem.

Theorem 3.2

If G is a graph with no isolated vertices, then $\mathcal{D}G$ is reconstructible from $\mathcal{D}'G$; it follows that G is edge-reconstructible if it is vertex-reconstructible. □

From this result we infer that every parameter or class of graphs which is vertex-reconstructible is also edge-reconstructible, provided that the graphs have no isolated vertices. For example, the connectivity κG of G is easily reconstructible from $\mathcal{D}G$. Hence to reconstruct κG from $\mathcal{D}'G$ for a graph with no isolated vertices we first obtain $\mathcal{D}G$ from which we readily determine κG.

In trying to show that a particular class $\mathcal{J}$ of graphs is vertex-reconstructible (or edge-reconstructible) the reconstructor is usually faced with a two-fold task: first he needs to tackle the problem of recognition, namely to recognize from $\mathcal{D}G$ (or $\mathcal{D}'G$) whether or not G is in the class $\mathcal{J}$. Having recognized this fact, he then proceeds to reconstruct the graph. Following Bondy and Hemminger in [BH1] we say that the class $\mathcal{J}$ is vertex-recognizable if, for each graph G in $\mathcal{J}$, every vertex-reconstruction of G is also in $\mathcal{J}$. We also say as in [BH1] that $\mathcal{J}$ is weakly vertex-reconstructible if, for each graph G in $\mathcal{J}$, all reconstructions of G that are in $\mathcal{J}$ are isomorphic to G. Hence, $\mathcal{J}$ is weakly vertex-reconstructible if, for each graph G in $\mathcal{J}$, one can reconstruct G uniquely when, apart from $\mathcal{D}G$, one is given the extra information that G is in $\mathcal{J}$. Clearly, the class $\mathcal{J}$ is vertex-reconstructible if and only if it is vertex-recognizable and weakly vertex-reconstructible. Edge-recognizable and weakly edge-reconstructible classes of graphs are defined in an analogous manner.

Various classes of graphs are known to be vertex-reconstructible or edge-reconstructible (see [BH1]). It has also been shown that for certain classes, like trees, not all the information in the vertex-deck is required for reconstruction. In the Appendix we shall give a result of this type for trees.

In most of this thesis we shall be primarily concerned with the reconstruction of certain classes of planar graphs. This, in fact, was one of the problems posed in [BH1], where it was suggested that one might try to reconstruct maximal planar graphs first. One class of graphs for which the Vertex-reconstruction Problem had been solved was the class of outerplanar graphs. In fact in this case, it was the class of maximal outerplanar graphs which was first reconstructed.

We now proceed to define some new ideas and prove some new results which we shall be referring to in later chapters on edge-reconstruction.

In various different contexts we shall be encountering the following situation. We are presented with a graph X in $\mathcal{D}'G$, and from the information we have at hand we know that there are at most two ways of reconstructing from X: namely as $X + e$ or as $X + f$. It follows that if G is not edge-reconstructible, then there is exactly one edge-reconstruction H not isomorphic to G, and $\{H,G\} = \{X+e,X+f\}$. This simple idea motivates the following definition.

Let G be not edge-reconstructible, and let there be exactly one edge-reconstruction H of G, $H \not\simeq G$. Let $X \in \mathcal{D}'G = \mathcal{D}'H$, and let $uv, vw \in EX^c$ be such that $G \simeq X + uv$ and $H \simeq X + vw$. We then say that G and H are associates with respect to $\{X, uv, vw\}$.

This definition of associates applies only in the context of edge-reconstruction. However, in the Appendix, where we consider a variant of the Vertex-reconstruction Problem for trees, a situation again arises where the concept of associates applies.

Another common situation we encounter is when a graph G turns out to be edge-reconstructible if it contains certain "configurations". In Chapter 7, where G is a 4-connected planar graph, wheel-sequences (which will be defined there) will be such configurations. We shall here define a more general type of configuration which is also applicable in the wider context of nonplanar graphs. This will then lead us to the definition of reconstructor sets and reconstructor sequences which will be of prime importance in Chapters 7, 9 and 10.

We define a valency-configuration to be a graph H whose vertices are assigned weights ωv, $v \in VH$. The graph G is said to contain the configuration H if H is a subgraph of G such that the weights ωv, for $v \in VH$, satisfy $\omega v = \rho_G v$.

Let $\mathfrak{J}$ be an edge-recognizable class of graphs. Let H be a valency-configuration which has the property that, for any $G \in \mathfrak{J}$, G is edge-reconstructible if G contains H. We then say that H reconstructs $\mathfrak{J}$. A reconstructor set for $\mathfrak{J}$ is a finite set of valency-configurations each of which reconstructs $\mathfrak{J}$.

In order to prove that a particular set of valency-configurations is a reconstructor set for $\mathfrak{J}$ we often employ a sequential argument as follows. We order the valency-configurations of the set in a sequence

$H_1, H_2, \ldots, H_t$, $t \geq 1$, in such a way that,

(i) H_1 reconstructs $\mathfrak{J}$,

and (ii) for each p, $1 < p \leq t$, if each one of $H_1, H_2, \ldots, H_{p-1}$ reconstructs $\mathfrak{J}$, then H_p also reconstructs $\mathfrak{J}$.

Then, the set of valency-configurations ordered in this way is called a reconstructor sequence for $\mathfrak{J}$. Clearly, ordering a set of valency-configurations as a reconstructor sequence amounts to proving that it is a reconstructor set.

REMARK. It is important to point out here that the definition of reconstructor sets and reconstructor sequences might be applied equally well to other types of configurations we may define apart from valency-configurations. In fact, in Chapter 7, we shall be dealing with reconstructor sets which contain wheel-sequences instead of valency-configurations.

We shall be using the above concepts primarily for the edge-reconstruction of planar graphs and for the edge-reconstruction of graphs with other topological properties. However, it is interesting to observe that these ideas of associates and reconstructor sequences have been independently used by Caunter [C1] and Swart [S2] in the different context of the edge-reconstruction of bidegreed graphs, although they do not use the terminology we have defined above.

We are indebted to Caunter for the reconstructor sequence in the next theorem. The proof we give here illustrates clearly the ideas of associates and reconstructor sequences.

Theorem 3.3

Let $\mathcal{J}$ be the class of graphs with minimum valency δ and maximum valency Δ. For $1 \le p \le \Delta - \delta + 1$, let the sequence (R_p) of valency-configurations be defined by $R_p = K_{1,p}$, with $VK_{1,p} = \{v\} \cup \{u_1, u_2, \ldots, u_p\}$ and $EK_{1,p} = \{vu_1, vu_2, \ldots, vu_p\}$, and whose weights are $\omega v = \delta + p - 1$, and $\omega u_i = \delta$, $i = 1, 2, \ldots, p$. Then the sequence (R_p) is a reconstructor sequence for $\mathcal{J}$.

Proof

First of all we note that the class $\mathcal{J}$ is edge-recognizable. Let $G \in \mathcal{J}$. If G contains R_1, then two δ-vertices are adjacent in G, so that clearly G is edge-reconstructible. Therefore R_1 reconstructs $\mathcal{J}$.

Now, let G contain R_2. We assume that G is not edge-reconstructible and derive a contradiction. Since G is not edge-reconstructible, then neither G nor any edge-reconstruction of G can contain configuration R_1. Hence, the only possible edge-reconstructions of G from $G - vu_1$ are G itself and $G_1 := G - vu_1 + u_1u_2$. Therefore G has only one edge-reconstruction H not isomorphic to it, and $H \simeq G_1$, that is, G and H are associates with respect to $\{G - vu_1, vu_1, u_1u_2\}$. Similarly we obtain that G and H are associates with respect to $\{G_1 - vu_2, vu_2, vu_1\}$, that is, if $G_2 := G_1 - vu_2 + vu_1$, then $G \simeq G_2$. Again we repeat the argument, giving that G and H are associates with respect to $\{G_2 - u_1u_2, u_1u_2, vu_2\}$, that is, if $G_3 := G_2 - u_1u_2 + vu_2$, then $H \simeq G_3$. However,

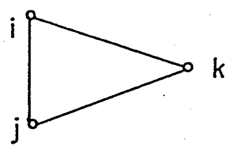
$$\begin{aligned} G_3 &= G_2 - u_1u_2 + vu_2 \\ &= G_1 - vu_2 + vu_1 - u_1u_2 + vu_2 \\ &= G - vu_1 + u_1u_2 - vu_2 + vu_1 - u_1u_2 + vu_2 \\ &= G. \end{aligned}$$

Therefore $H \simeq G$, giving the required contradiction. Hence we have shown that R_2 reconstructs $\mathcal{J}$.

Now, let p be such that $3 \le p \le \Delta - \delta + 1$, let each of $R_1,\ldots,R_{p-1}$ reconstruct $\mathcal{J}$, and let $G \in \mathcal{J}$ contain R_p. Let us assume that G is not edge-reconstructible, and that H is an edge-reconstruction of G, $H \not\cong G$. Therefore if we reconstruct from $G - vu_p$ we obtain that $H \cong G - vu_p + ab$, where $\{a,b\} \neq \{v,u_p\}$. But since the minimum valency of H is δ, we may assume that $a = u_p$. Moreover, since $\{\{\rho_G v, \rho_G u_p\}\} = \{\{\rho_H a, \rho_H b\}\}$, and since $\rho_G v = \delta + p - 1 > \delta + 1$, then $b \neq u_i$, for any $i = 1,2,\ldots,p-1$. Therefore H contains configuration R_{p-1}, implying that H is edge-reconstructible, since R_{p-1} reconstructs $\mathcal{J}$. But this is a contradiction. Therefore G is edge-reconstructible, that is, R_p reconstructs $\mathcal{J}$. It follows by induction that the sequence (R_p), $1 \le p \le \Delta - \delta + 1$, is a reconstructor sequence for $\mathcal{J}$. □

REMARK. The type of argument we have just employed to show that R_2 reconstructs $\mathcal{J}$ will be of great importance in Lemmas 7.3 and 7.4 which are crucial results for Chapter 7.

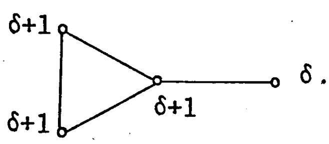
Before proving the last result of this chapter we give one final definition. An <u>(i,j,k)-triangle</u> is the valency configuration



that is, a 3-circuit whose vertices have weights i, j, k respectively.

<u>Theorem 3.4</u>

Let $\mathcal{J}$ be the class of graphs with minimum valency δ and maximum valency at least $\delta + 1$. Then the sequence (S_1, S_2, S_3, S_4) is a reconstructor sequence for $\mathcal{J}$, where S_1 and S_2 are the same as R_1 and R_2 respectively of Theorem 3.3, S_3 is a $(\delta,\delta+1,\delta+1)$-triangle and S_4 is the valency-configuration



Proof

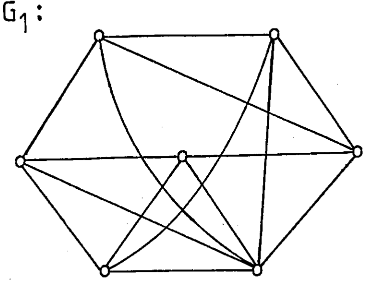
Let G be any graph in $\mathcal{J}$ and let us assume that G contains S_3. Let the $(\delta+1)$-vertices of S_3 be u and v, and the δ-vertex be w. Let us assume that G is not edge-reconstructible, and that $H \neq G$ is an edge-reconstruction of G. Therefore if we reconstruct from $G - vw$, we obtain that $H \simeq G - vw + wx$ for some vertex $x \neq v$. But then H contains the configuration S_2, giving that H is edge-reconstructible, a contradiction. Hence S_3 reconstructs $\mathcal{J}$.

Now, let G contain S_4, and let us assume that it is not edge-reconstructible. Let u be the δ-vertex of S_4, let v be the vertex adjacent to u in S_4, and let a, b be the other two vertices of S_4. Furthermore, let $H \neq G$ be an edge-reconstruction of G. Then, reconstructing from $G - uv$ we obtain that $H \simeq G - uv + ux$ for some vertex $x \neq v$. Moreover, since $\rho_G v = \rho_H x$, then x is equal to neither a nor b. Therefore H contains S_3, again giving the contradiction that H is edge-reconstructible. Hence S_4 reconstructs $\mathcal{J}$. □

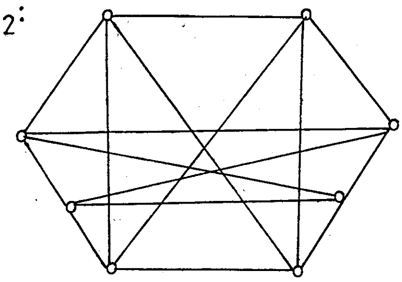
PART II VERTEX-RECONSTRUCTION

In this part we show that maximal planar graphs are vertex-reconstructible. This work was started by Fiorini and Manvel who solved the problem for maximal planar graphs with minimum valency greater than 3. In Chapter 4 we are concerned with the problem of vertex-recognition, whereas in Chapter 5 we deal with vertex-reconstruction. The main problem which we face in Chapter 5 is that the vertex-deleted subgraphs which we are using for reconstruction have non-equivalent k-representations. Hence, most of this chapter is devoted to a study of these graphs and to how non-equivalent k-representations of a k-representable graph are related.

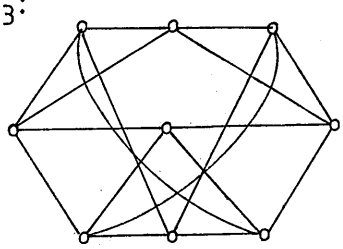
G_1:



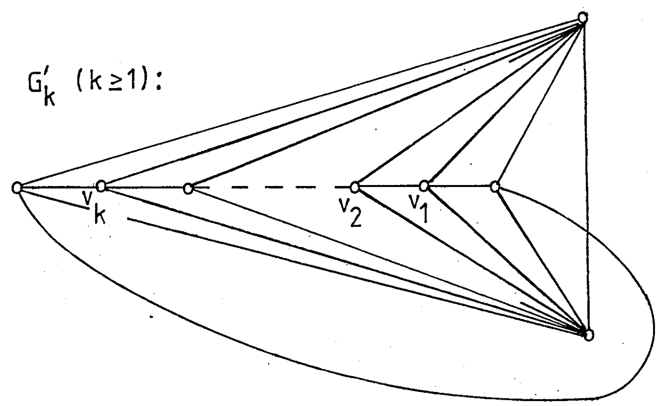
G_2:



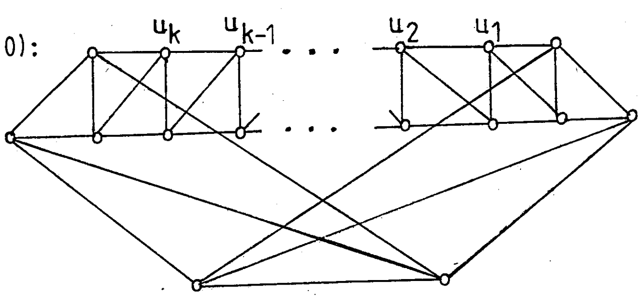
G_3:



G'_k $(k \geq 1)$:



G''_k $(k \geq 0)$:



The set of graphs Ω shown in Figure 1 of [FM1] (see Theorem 4.2 on the next page).

CHAPTER 4 MAXIMAL PLANAR GRAPHS: VERTEX-RECOGNITION

In this and the next chapter we shall complete the work started in [F2] and [FM1] on the vertex-reconstruction of maximal planar graphs. In [F2] and [FM1] it was shown that maximal planar graphs with minimum valency at least 4 are vertex-reconstructible, so that there remains to show that maximal planar graphs with minimum valency 3 are also vertex-reconstructible. In this chapter we shall be concerned with the problem of vertex-recognition.

MAIN THEOREM OF CHAPTER 4

Maximal planar graphs are vertex-recognizable.

Clearly, if some G_v in $\mathcal{D}G$ is not planar, then neither is G. But what can we say about the converse? If every G_v is planar, then is G itself necessarily planar? This problem was first tackled in [F2] where it was shown that,

Theorem 4.1

A graph G with minimum valency 5 is planar if and only if every G_v in $\mathcal{D}G$ is planar. □

Although this result is no longer true if the restriction on the minimum valency is lifted, the following theorem [FM1] characterizes all critical graphs whose minimum valency is at least 4. †

Theorem 4.2

Let Ω be the set of graphs shown in Figure 1 of [FM1]. Then a graph G with minimum valency at least 4 is critical if and only if $G \in \Omega$. □

† Actually, critical graphs were first studied in [W2]. However, it is not immediately clear from the characterization given in [W2] which are the critical graphs in which we are interested for the purpose of reconstruction.

Actually, Theorems 4.1 and 4.2 give more than the vertex-recognition of maximal planar graphs. Since all the graphs in Ω are vertex-reconstructible, then these theorems imply that the class of all planar graphs with minimum valency at least 4 is vertex-recognizable. Since a planar graph G is maximal planar if and only if $\varepsilon G = 3 \cdot \nu G - 6$, it then follows that, if G has minimum valency at least 4, we can determine from $\mathcal{D}G$ whether or not G is maximal planar.

The proofs of Theorems 4.1 and 4.2 are long and involved, and it seems even more difficult to show that the class of all planar graphs with minimum valency 3 is vertex-recognizable. The problem is made easier by restricting ourselves to showing that the class of maximal planar graphs with minimum valency 3 is vertex-recognizable; we solve this problem in Theorems 4.4 and 4.5. However, to make our treatment self-contained we first prove in Theorem 4.3, independently of Theorems 4.1 and 4.2, that maximal planar graphs with minimum valency at least 4 are vertex-recognizable. We first give a few definitions.

If the graph H is a subdivision of either K_5, K_4 or $K_{3,3}$, we call a vertex of H a <u>minor vertex</u> if it has valency 2 and a <u>major vertex</u> otherwise. If H is a subdivision of K_n, $n = 4$ or 5, and the major vertices are labelled $1, 2, \ldots, n$, we sometimes say that H is K_n, and we write $H = K(1,2,\ldots,n)$. If H is a subdivision of $K_{3,3}$, and the major vertices are 1, 2, 3 and a, b, c, we often say that H is $K_{3,3}$, and we write $H = K(1,2,3;a,b,c)$. If x and y are two major vertices of H, then the chain $C[x,y]$ which contains no other major vertex is called a <u>primary chain</u> of H.

We shall also need the following lemma.

Lemma 4.1

Let $C = v_1v_2\dots v_tv_1$ be a circuit in a plane graph G such that $\mathrm{Int}C$ (or $\mathrm{Ext}C$) contains no vertex of G and is triangulated (that is, all the faces of G in $\mathrm{Int}C$ (or $\mathrm{Ext}C$) are 3-faces). If v_t is not adjacent to v_i, $i \in \{2,\dots,t-2\}$, then v_1 is adjacent to v_{t-1}.

Proof

We may assume that $\mathrm{Int}C$ is triangulated. Let H be that subgraph of G induced by EC and all the edges of G embedded in $\mathrm{Int}C$, and let H' be the graph obtained from H by adding an extra vertex v and joining it to all the vertices of H. Then H' is maximal planar, and the neighbours of v_t in H' are v, v_1 and v_{t-1}. Therefore v_1 is adjacent to v_{t-1} in H' and hence in G. □

Theorem 4.3

Let G be a 3-connected graph with minimum valency at least 4. Then G is maximal planar if and only if each G_v has a ρv-representation.

Proof

The necessity of the condition is obvious. To prove sufficiency, we let G be a 3-connected graph of minimum valency at least 4 each of whose vertex-deleted subgraphs G_v has a ρv-representation. By Theorem 2.3, there exists a vertex v such that G_v is 3-connected and hence has a unique plane representation by Theorem 2.7. We shall therefore henceforth assume that we are dealing with the plane representation of G_v. If v is adjacent to all the vertices on the ρv-face of G_v, then there is nothing to prove. We shall therefore assume that v is adjacent to a vertex w not on the ρv-face of G_v, so that w is incident solely to 3-faces in G_v. The aim of the following is to show that this assumption on v leads to a contradiction.

Let $\{w_1,w_2,\ldots,w_r\}$ be the neighbours of w in G_v. Thus, the subgraph of G induced by $\{w_1,w_2,\ldots,w_r\}$ is Hamiltonian. We shall first show that v must be adjacent to some vertex which is not a neighbour of w. Let us assume that v is adjacent only to neighbours of w (apart from w itself). Then, since $\rho v \geq 4$ in G, there must be at least three vertices w_m, w_n, w_o adjacent to v. Therefore the nonplanar graph of Figure 4.1 is a subgraph of G, and hence it contains all the vertices of G. (In this and similar figures, solid lines indicate edges and dashed lines indicate chains which could possibly be edges.) But then $\rho w = r + 1$ in G, so that G_w is a graph on $r + 1$ vertices and which has an $(r+1)$-representation. Thus, G_w is outerplanar. But this is impossible since G_w contains $K(w_m,w_n,w_o,v)$, a subdivision of K_4. It follows that v is adjacent to some vertex t not in $\{w_1,w_2,\ldots,w_r\}$.

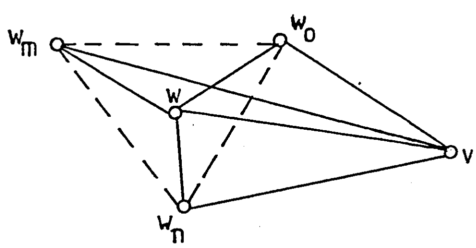


Figure 4.1

Now, G_v is 3-connected, so that there exist three internally disjoint chains from t to w. We deduce that there exist three internally disjoint chains $C[t,w_a]$, $C[t,w_b]$ and $C[t,w_c]$ in G_v, (where a,b,c are distinct elements of $\{1,2,\ldots,r\}$) such that none of these chains contains w. We now consider two distinct cases.

Case 1 $r = 3$

In this case, the nonplanar graph in Figure 4.2 is a subgraph of G, and hence contains all the vertices of G. Now, we cannot have that all the three chains $C[t,w_x]$, $x = 1,2,3$, are edges; otherwise G_v would be maximal planar, so that G_v would have no ρv-representation.

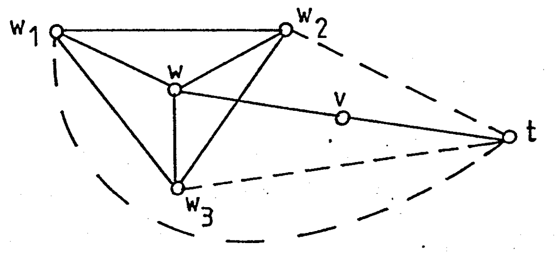


Figure 4.2

We therefore consider the following subcases.

Case 1.1 Exactly two of $C[t,w_x]$, $x = 1,2,3$, are edges

We can assume without loss of generality that $C[t,w_3]$ is not an edge. Let $C[t,w_3]$ be $w_3s_1s_2\ldots s_kt$. Then v cannot be adjacent to w_3 or to any of $s_1,s_2,\ldots,s_{k-1}$, because otherwise $G - s_k$ would contain $K_{3,3} = K(w,w_3,t;w_2,v,w_1)$.

Now, the ρv-face of G_v must lie either in the interior of the circuit $w_2w_3s_1s_2\ldots s_ktw_2$ or the interior of $w_1w_3s_1s_2\ldots s_ktw_1$ (see Figure 4.3).

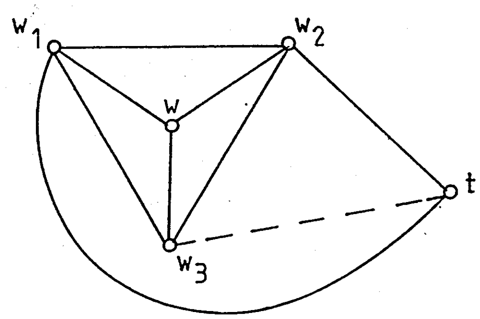


Figure 4.3

We can assume without loss of generality that the ρv-face is in the interior of $w_2w_3s_1\ldots s_ktw_2$. Then w_1 is adjacent to each s_j for all j. Now, each vertex s_j has valency at least 4 in G and, moreover, no s_j can be adjacent to s_i $(|i-j| \geq 2)$, since otherwise there exists

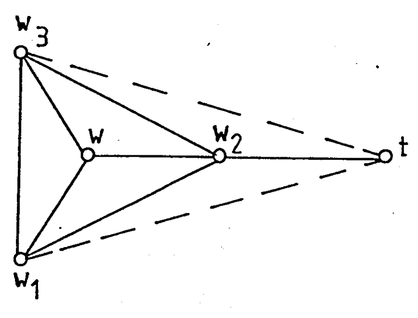
h $(i < h < j$ or $j < h < i)$ such that $G - s_h$ contains $K_5 = K(t,w,w_1,w_2,w_3)$. By a similar argument, s_j cannot be adjacent to any of t, w, w_3. Since s_j has valency at least 4, we conclude that w_2 is adjacent to each s_j $(j < k)$. Also, s_k cannot be adjacent to w_2, since otherwise G_v would be maximal planar and hence would have no ρv-representation. Thus, s_k must be adjacent to v.

Now, v has valency at least 4, so that v must be adjacent to at least one of w_1, w_2. However, if v is adjacent to w_1, then $G - w_1$ is a graph on $5 + k$ vertices and has a $(5+k)$-representation. Hence $G - w_1$ is outerplanar, which is impossible since it contains $K(v,t,w,s_k)$. We conclude that v is adjacent to w_2. But this is contradictory since G_t then contains $K(v,w_1,w_3;w,w_2,s_k)$.

<u>Case 1.2</u> <u>Only one of $C[t,w_x]$, $x = 1,2,3$, is an edge</u>

We assume that $C[t,w_1] = w_1q_1q_2\ldots q_ht$ and $C[t,w_3] = w_3s_1s_2\ldots s_kt$ are not edges . Then, as in Case 1.1, v cannot be adjacent to any of w_1, w_3, $s_1,\ldots,s_{k-1}$, $q_1,\ldots,$ q_{h-1}.

Now, the interior of either $w_2w_3s_1\ldots s_ktw_2$ or $w_2w_1q_1\ldots q_ktw_2$ must be triangulated in G_v (see Figure 4.4). If both are triangulated,



<u>Figure 4.4</u>

then w_2 is adjacent to each s_i and to each q_i, so that v cannot be adjacent to w_2; otherwise (as in Case 1.1), $G - w_2$ is both outerplanar and contains $K(t,w,w_1,w_3)$. Thus, v must be adjacent to both s_k and q_h. But then G_t contains $K(v,w_1,w_2;w,w_3,q_h)$.

So we can assume that only the interior of $w_2w_1q_1\ldots q_htw_2$ is triangulated, so that w_2 is adjacent to each q_j. Hence we again have that v is not adjacent to *both* s_k and q_h; otherwise, as in the previous case, G_t contains $K(v,w_1,w_2;w,w_3,q_h)$. Thus v is adjacent to w_2 and to one of s_k or q_h; say v is adjacent to s_k. Since $\text{Int}(w_2w_3s_1\ldots tw_2)$ contains the ρv-face of G_v, then $\text{Ext}(w_1q_1\ldots ts_k\ldots w_3w_1)$ must be triangulated, so that s_k is adjacent to q_h, by Lemma 4.1. We then deduce that G_t contains $K(s_k,w,w_2;v,w_1,w_3)$, as in Figure 4.5, a contradiction.

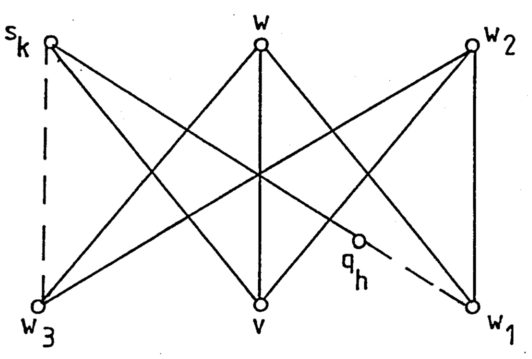


Figure 4.5

Case 1.3 None of $C[t,w_x]$, $x = 1,2,3$, is an edge

Let $C[t,w_1] = w_1q_1q_2\ldots q_ht$, $C[t,w_2] = w_2p_1p_2\ldots p_jt$, and $C[t,w_3] = w_3s_1s_2\ldots s_kt$. As before, we have that v cannot be adjacent to any vertex in $C[w_1,q_{h-1}]$, $C[w_2,p_{j-1}]$, $C[w_3,s_{k-1}]$, and that v is adjacent to at least two of q_h, p_j, s_k; say v is adjacent to s_k and q_h.

Now, in Figure 4.6, at least one of $\text{Int}(w_3s_1\ldots tp_j\ldots w_2w_3)$ or $\text{Int}(w_1q_1\ldots tp_j\ldots w_2w_3)$ is triangulated; say $\text{Int}(w_3s_1\ldots tp_j\ldots w_2)$ is. Then by Lemma 4.1, s_k is adjacent to p_j, so that G_t contains $K(w,w_1,s_k;v,w_2,w_3)$, as in Figure 4.6. This completes Case 1.

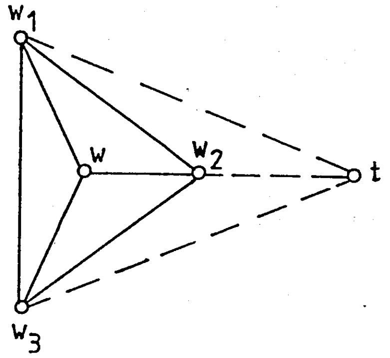


Figure 4.6

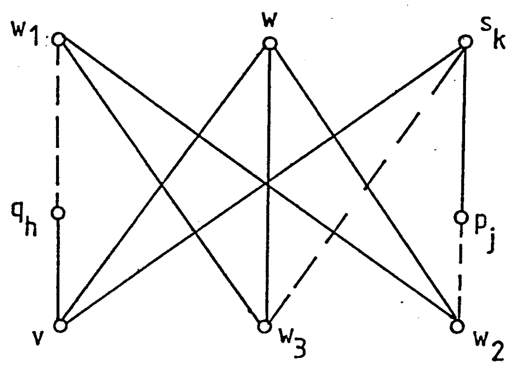


Figure 4.7

Case 2 $r \geq 4$

In this case, referring to Figure 4.8, at least one of $C[w_x,w_y]$ $\{x,y\} \subset \{a,b,c\}$ is not an edge. We assume that $C[w_a,w_c]$ is not an edge, and we let $p \in C]w_a,w_c[$.

If v is adjacent to a vertex on either $C]w_a,w_b]$, $C]w_c,w_b]$ or $C]t,w_b]$, then G_p has a subgraph contractible to $K(w_a,v,w_c;w,t,w_b)$. Moreover, v cannot be adjacent to any vertex of $C]p,w_a]$ or $C]p,w_c]$; otherwise, $G - w_b$ would contain a subgraph contractible to $K(v,w_a,w_c;p,t,w)$. We conclude that v can only be adjacent to vertices in $C]t,w_a]$ or $C]t,w_c]$, apart from t and w. Furthermore, v cannot be adjacent to two vertices both from $C]t,w_a]$ or both from $C]t,w_c]$; otherwise, assuming that v is adjacent to vertices s and q on

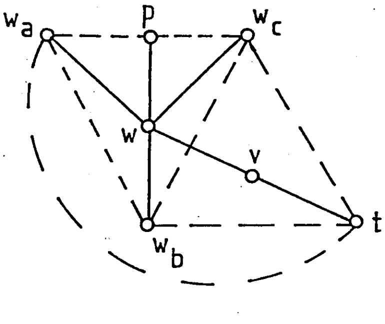


Figure 4.8

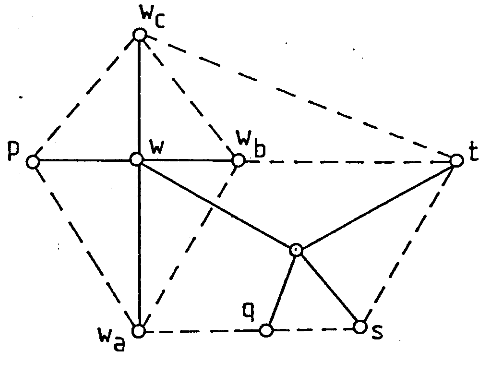


Figure 4.9

$C]t,w_a]$, say (see Figure 4.9), then G_s would contain $K(w,w_a,w_c;p,v,w_b)$. Thus, v must be adjacent to a vertex q on $C]t,w_a]$ and to a vertex s on $C]t,w_c]$. But then G_t contains $K(w,w_a,w_c;p,v,w_b)$.

Since all cases lead to a contradiction, the proof is complete. □

We now turn our attention to maximal planar graphs with minimum valency 3.

Theorem 4.4

If G is a graph which has at least two vertices of minimum valency 3 and whose order is at least 7, then G is maximal planar if and only if (i) $\varepsilon G = 3 \cdot \nu G - 6$, and (ii) each G_v is planar.

Proof

The necessity of the condition is obvious. To prove sufficiency, let G be a graph of order ν, having $3\nu - 6$ edges, and each of whose vertex-deleted subgraphs is planar. Let w be a 3-vertex. Then

$$\nu(G_w) = \nu G - 1$$

$$\text{and} \qquad \varepsilon(G_w) = \varepsilon G - 3 = 3 \cdot \nu G - 9 = 3 \cdot \nu(G_w) - 6$$

so that G_w is maximal planar. ... ... (1)

Let x, y, z be the neighbours of w. If x say, has valency 3 in G, then x has valency 2 in G_w, which contradicts (1). Thus, each of x, y, z has valency at least 4 in G. ... ... (2)

Let v be a 3-vertex other than w. (The vertex v exists by hypothesis, and v is not a neighbour of w, by (2).) The graph G_v is maximal planar, by (1), so that x, y, z induce a 3-circuit C in G_v, and hence in G and in G_w. Let G_w be embedded in the plane (we recall that G_w, being maximal planar, is 3-connected by Corollary 2.1, and so has a unique plane representation by Theorem 2.7). We can assume that C does not bound a face in the plane representation of G_w; otherwise, it follows immediately that G is maximal planar. Thus, we can assume that there are vertices of G_w both in ExtC and in IntC when G_w is embedded in the plane. We shall show that this assumption leads to a contradiction, so that C does in fact bound a face of G_w.

Now, the neighbours of v induce a 3-circuit C' in G_w. We want to show first that $C' = C$. Since G_w is planar, then either

$$C' \subset \overline{\text{IntC}} = C \cup \text{IntC}$$

$$\text{or else} \quad C' \subset \overline{\text{ExtC}} = C \cup \text{ExtC}.$$

Without loss of generality, we can assume that $C' \subset \overline{\text{IntC}}$. Let us suppose that $C' \neq C$ and that $k \in VC' - VC$, so that k is in IntC. Let h be any vertex in ExtC. Since $G_w - v$ is maximal planar, and hence 3-connected, then there exist three internally disjoint chains $C_i[h,k]$ $(i = 1,2,3)$, from h to k, by Theorem 2.2. But since the set $\{x,y,z\}$ separates h and k in $G_w - v$, then we may assume that $x \in V(C_1[h,k])$, $y \in V(C_2[h,k])$ and $z \in V(C_3[h,k])$. But then, G_v contains the subgraph of Figure 4.10. Thus, G_v is nonplanar, which is impossible. We therefore conclude that $C' = C$, that is, any 3-vertex of G is adjacent to x, y and z.

Now, let us assume first that G has at least three vertices u, v, w of valency 3. We let p be any vertex not in the set $\{u,v,w,x,y,z\}$; p exists since $\nu G \geq 7$. Therefore G_p contains $K(u,v,w;x,y,z)$, so that it is nonplanar, contrary to our assumptions. We therefore have to consider the remaining case when G has exactly two vertices v and w of valency 3. Since G_w is a maximal planar graph with at least 6 vertices, then x, y, z (the neighbours of v) must have valency at least 4 in G_w. Thus, G_w has exactly one 3-vertex, namely v.

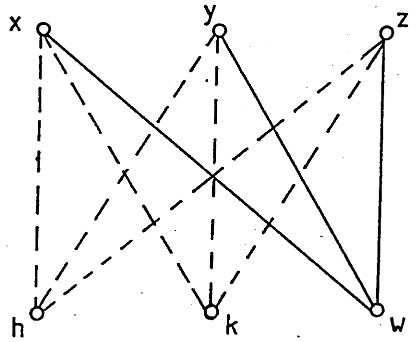


Figure 4.10

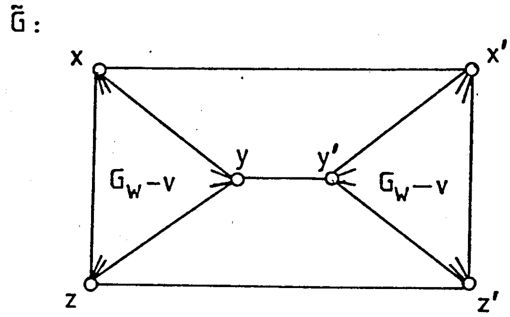


Figure 4.11

We now want to show that there exists a vertex $q \in VG_w - \{v,x,y,z\}$, such that $G_w - q$ is still 3-connected. We construct a new graph $\tilde{G}$ by taking two copies of $G_w - v$ labelled $x,y,z,\ldots$ and $x',y',z',\ldots$ respectively, and joining x to x', y to y', and z to z' by three independent edges, as in Figure 4.11. We note that the minimum valency of $\tilde{G}$ is at least 4 and that $\tilde{G}$ is 3-connected. We now apply Theorem 2.3 to $\tilde{G}$ to find a vertex q whose deletion from $\tilde{G}$ results in a 3-connected graph. By our construction of $\tilde{G}$, the vertex q cannot be any of x, y, z, x', y', z', so that q must be be in IntC, as required. Clearly, $G_w - q$ is 3-connected.

Since in G_w we can find a vertex $p \notin \{v,x,y,z,q\}$, then applying Theorem 2.2 to the 3-connected graph $G_w - q$, we see that there exist

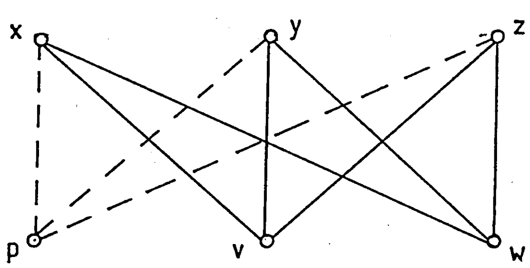


Figure 4.12

three internally disjoint chains C[p,x], C[p,y], C[p,z], as before. Thus, G_q contains the graph K(p,v,w;x,y,z) of Figure 4.12. This final contradiction completes the proof. □

Since all graphs with at most nine vertices are vertex-reconstructible (see [BH1]), and since the number of edges and the valency list of G can be determined from *DG*, this theorem gives the vertex-recognition of maximal planar graphs with at least two 3-vertices. We now show that maximal planar graphs with a unique 3-vertex are vertex-recognizable. We shall need the following lemma.

Lemma 4.2

Let G be a graph with a unique 3-vertex v, and such that Nv induces a 3-circuit. Then G cannot be critical.

Proof

Let us suppose that G is critical. Then G contains a subgraph H, with VG = VH, and which is a subdivision of either K_5 or $K_{3,3}$. If H is K_5, then v must be a minor vertex with neighbours x and y (say) in H. But then xy is an edge of G, so that G_v still contains a subdivision of K_5, a contradiction. If H is $K_{3,3}$ and v a minor vertex of H, then the same argument applies. Therefore let v be a major vertex of H. If one of the three primary chains of H containing v is not an edge, then G_v contains a subdivision of $K_{3,3}$. We may therefore assume that G contains the subgraph of Figure 4.13.

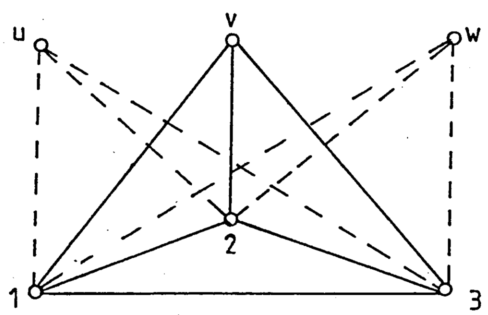


Figure 4.13

Now, $\rho_G u \geq 4$, by hypothesis, so that u must be adjacent to a fourth vertex z which cannot lie on a primary chain of H joining u to 1, 2 or 3, since otherwise there would exist a vertex t in $C]u,z[$ for which G_t contains a subdivision of $K_{3,3}$. But then z is on a primary chain of H joining w to 1, 2 or 3, so that G_v either contains a subdivision of K_5 (if $z = w$) or a subgraph contractible to K_5. Hence G cannot be critical. □

<u>Lemma 4.3</u>

Let G be a graph with a unique 3-vertex v_0 and such that $\varepsilon G = 3 \cdot \nu G - 6$. Then G is maximal planar if and only if (i) every G_v is planar, and (ii) $N_G v_0$ induces a 3-circuit in G.

<u>Proof</u>

The necessity of the condition is clear. To see the converse, we observe that, by (i) and (ii) and by Lemma 4.2, G is planar. But $\varepsilon G = 3 \cdot \nu G - 6$, therefore G is maximal planar. □

<u>Theorem 4.5</u>

Maximal planar graphs with a unique 3-vertex are vertex-recognizable.

<u>Proof</u>

The number of edges of a graph G and the uniqueness of the 3-vertex v_0 can all be determined from G. Therefore in view of Lemma 4.3, to show that we can determine from $\mathcal{D}G$ whether or not G is maximal planar, all we have to do is to prove that we can determine whether or not $N_G v_0$ induces a 3-circuit in G.

Since for any G_v in $\mathcal{D}G$ we can determine $\rho_G v$, then we can identify the graph $G - v_0$. By Kelly's Lemma, we can determine whether or not v_0 is contained in a subgraph of G isomorphic to K_4. But $N_G v_0$ induces a 3-circuit in G if and only if v_0 is contained in such a subgraph, and hence we can determine from DG whether or not $N_G v_0$ induces a 3-circuit in G. The proof of the theorem is thus complete. □

This final result concludes the proof of the Main Theorem of this chapter.

CHAPTER 5 MAXIMAL PLANAR GRAPHS: VERTEX-RECONSTRUCTION

After having proved in Chapter 4 that maximal planar graphs are vertex-recognizable we now prove further that this class of graphs is indeed vertex-reconstructible.

MAIN THEOREM OF CHAPTER 5

Maximal planar graphs are vertex-reconstructible.

We first recall that the following result was proved in [FM1]. We reproduce the proof for completeness' sake.

Theorem 5.1

Every maximal planar graph whose minimum valency is at least 4 is vertex-reconstructible.

Proof

If G is a maximal planar graph whose minimum valency is at least 4, then its minimum valency is reconstructible from the deck $\mathcal{D}G$ of vertex-deleted subgraphs of G, as is the fact that G is maximal planar. Now, since every maximal planar graph is 3-connected, and since $\delta G \geq 4 = \frac{1}{2}(3 \cdot 3 - 1)$, it follows from Theorem 2.3 that in G there is a vertex v_0 such that $G - v_0$ is 3-connected. Therefore $G - v_0$ has a unique ρv_0-representation. But then there is a unique way of reconstructing G from $G - v_0$, namely by joining the vertex v_0 to the ρv_0 vertices incident to the unique ρv_0-face of G. $\square$

In view of this result, there remains to show that maximal planar graphs of minimum valency 3 are also vertex-reconstructible. The method used in the above proof unfortunately fails, since we are now no longer assured that G has a vertex v such that $\rho v \geq 4$ and G_v is 3-connected. We therefore have to introduce further concepts.

We first define an ordinary vertex to be a vertex whose valency is at least 4. Now, given a maximal planar graph G of minimum valency 3, we can recognize the maximal planarity of G from the vertex-deck of G, as seen in Chapter 4. Therefore for any ordinary vertex v, we need only consider the ρv-representations of G_v, any vertex-reconstruction of G being obtained from some G_v by adding a vertex and joining it to the vertices incident to the ρv-face of a ρv-representation of G_v. For this reason, Section 5.1 deals with k-representable graphs. Moreover, if for some ordinary vertex w of G, G_w has a unique ρw-representation, then G is uniquely vertex-reconstructible from G_w. We can therefore assume that for any ordinary vertex w of G, G_w has at least two non-equivalent ρw-representations. Such a maximal planar graph will be called a collapsible graph. These graphs will be studied in Section 5.2. If F is a face of a plane graph and $v_0 v_1 \ldots v_{k-1} v_0$ is the boundary circuit of F, we shall often say, for convenience of notation, that F is the face $v_0 v_1 \ldots v_{k-1} v_0$, provided this does not give rise to ambiguity.

SECTION 5.1 - PROPERTIES OF k-REPRESENTABLE GRAPHS

Let R be a plane representation of a graph G, and let C be a boundary circuit of a face in R. Then by a cyclic labelling $c_0, c_1, \ldots, c_{r-1}$ of the vertices of C we mean a labelling in the order in which they appear in R, that is, for $i = 0,1,\ldots,r-1$, $c_i c_{i+1}$ (modulo r) are edges of C.

Lemma 5.1

Let R be a plane representation of a graph G, and let C be a circuit bounding a face in R, and $c_0, c_1, \ldots, c_{r-1}$ a cyclic labelling of C. Let R' be another plane representation of G, such that the vertices of C form a circuit C' which bounds an r-face in R'. Then the vertices of C appear on C' in the same cyclic order as in R, that is, $C' = C$.

Proof

If $r = 3$, then the result is trivially true. We can therefore assume that $r > 3$. For a contradiction we assume that the lemma is false. Therefore there exists a vertex c_i such that c_{i-1} or c_{i+1} is not adjacent to c_i in C'. We can therefore assume, without loss of generality, that c_1 is not adjacent to c_2 in C'. It follows that there exists a j, such that $2 < j < r$, and c_j is adjacent to c_1 in C'. Hence the edge c_1c_j is a chord for C.

Now, there must exist an edge c_kc_t of C' such that

$$1 < k < j < t \leq r - 1$$

because otherwise $C' - c_1c_j$ would be disconnected, which is impossible. But then the edge c_kc_t is also a chord for the circuit C, and the pair of chords c_1c_j and c_kc_t, regarded as bridges of C, are skew.

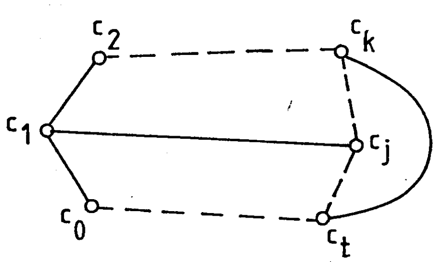


Figure 5.1

Therefore by Theorems 2.5.1 and 2.5.3 of [O1], the circuit C can never be the boundary of a face of G (see Figure 5.1). This contradiction completes the proof of the lemma. □

Lemma 5.2

Let G be a k-representable graph, and let R be a k-representation of G such that C is the k-circuit bounding the k-face in R. If R' is another k-representation of G such that C also bounds the k-face in R', then R is equivalent to R'.

Proof

Let ψ be the identity isomorphism on G. Let H be obtained from R by adding the vertex w inside the k-face of R and joining it to each of the vertices of C. Let H' be similarly obtained from R' by the addition of the vertex w'. Then ψ can be extended to an isomorphism $\psi' : VH \to VH'$ such that $\psi' w = w'$. But H and H' are plane representations of a maximal planar (and hence 3-connected) graph. Therefore by Theorem 2.7, ψ' maps boundary circuits of faces into boundary circuits of faces. Hence, a circuit Γ bounds a face in R if and only if $\psi\Gamma$ bounds a face in R', that is, R is equivalent to R'. □

Thus we see that if R is a k-representation of G, and C the circuit bounding the k-face of R, then any other k-representation R' not equivalent to R must have some vertex or vertices not in VC incident to its k-face.

Now, let G be a k-representable graph, and let R be a k-representation of G. Let C be the circuit bounding the k-face of R, C being labelled in the cyclic order $c_0, c_1, \ldots, c_{k-1}$. Let $c_i c_{i+1} c_{i+2} c_i$ be a separating triangle for G. Let $T = c_i c_{i+1} c_{i+2} c_i$, and let the components of $G - T$ be H_T and K_T, where H_T is defined as that component of $G - T$ which has some vertex adjacent to c_{i+1} in G. Then $\overline{H}_T = \langle V(H_T) \cup VT \rangle$ is a maximal planar graph. Let $c_i y c_{i+2} c_i$ be the face of $\overline{H}_T$, different from $c_i c_{i+1} c_{i+2} c_i$, which is incident to the edge $c_i c_{i+2}$ in $\overline{H}_T$.

We now observe that H_T is a bridge of the circuit C. Moreover, if this bridge is transferred to IntC (or to ExtC if the k-face of R is the unbounded face), we then obtain another k-representation of G, with the vertex y on the k-face instead of c_{i+1}. We call such a bridge in a k-representation, with three attachments c_i, c_{i+1}, c_{i+2} forming a separating triangle, an <u>arch</u>; the maximal planar graph $\overline{H}_T$ with the three vertices c_i, c_{i+1}, c_{i+2} so labelled, we call a <u>span</u>. Since T is a separating triangle, then the order of $\overline{H}_T$ is greater than 3. Therefore each of c_i, c_{i+1}, c_{i+2} has valency at least 3 in $\overline{H}_T$. We shall call these three vertices the primary vertices of the span $\overline{H}_T$, and c_i, c_{i+2} will be called the pivots of the span. The vertices c_{i+1} and y will be called the replaced vertex and the replacement vertex respectively. We emphasize that a span is a maximal planar graph with a labelled boundary circuit of a face, and with one of the three labelled vertices designated as the replaced vertex. In general we shall adopt the notation $S(abc)$ or $S(cba)$ to denote a span with a and c as pivots and b as replaced vertex.

We have seen above that if R is a k-representation of a graph G, then an arch transfer also gives another k-representation. Now, Ore has shown (Theorem 2.5.4 in [01]) that if R and R' are two plane representations of a 2-connected graph G, then R can be transformed into a representation equivalent to R' by a sequence of bridge transfers. We shall now show that if R and R' are two k-representations of G, then R can be transformed into a representation equivalent to R' by a sequence of arch transfers.

Lemma 5.3

Let G be a k-representable graph, and let R be a k-representation of G. Let C be the circuit bounding the k-face in R and labelled cyclically as $c_0c_1c_2\ldots c_{k-1}c_0$. Let R' be another k-representation of G such that the vertex c_{i+1} is not incident to the k-face of R'. Then we have:

(i) $\rho(c_{i+1}) > 2$,

and (ii) c_i is adjacent to c_{i+2}.

Proof

If $\rho(c_{i+1}) = 2$, then adding a vertex w inside the k-face of R' and joining it to each vertex incident to the k-face of R' gives a maximal planar graph of order greater than 3, and with a vertex of valency 2, a contradiction. Therefore $\rho(c_{i+1}) > 2$.

Now, since in R' the vertex c_{i+1} is not incident to the k-face, then it must be incident only to 3-faces. Therefore the subgraph of G induced by Nc_{i+1} is Hamiltonian. In R, all the faces which are incident to c_{i+1}, except one, are 3-faces (see Figure 5.2).

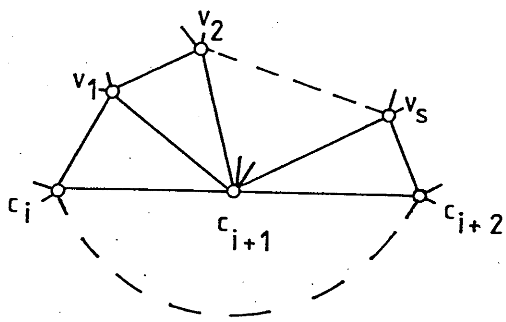


Figure 5.2

We can therefore label the neighbours of c_{i+1} in order, starting from c_i to c_{i+2} as they appear in R, giving that $Nc_{i+1} = \{c_i, v_1, v_2, \ldots, v_s, c_{i+2}\}$ (see Figure 5.2).

If $\rho(c_{i+1}) = 3$, then the fact that the subgraph induced by Nc_{i+1} is Hamiltonian implies that c_{i+2} is adjacent to c_i. We may therefore assume that $\rho(c_{i+1}) > 3$.

Now, let us assume that c_i is not adjacent to c_{i+2}. Since in R' the vertex c_{i+1} is incident only to 3-faces, then c_i must be adjacent to some other vertex from Nc_{i+1}, apart from v_1. Let $j = \text{maximum}\{r: v_r \text{ adjacent to } c_i\}$. Thus c_{i+2} cannot be adjacent to v_t, for $t < j$, because otherwise G would contain the graph of Figure 5.3 which is a subdivision of K_5.

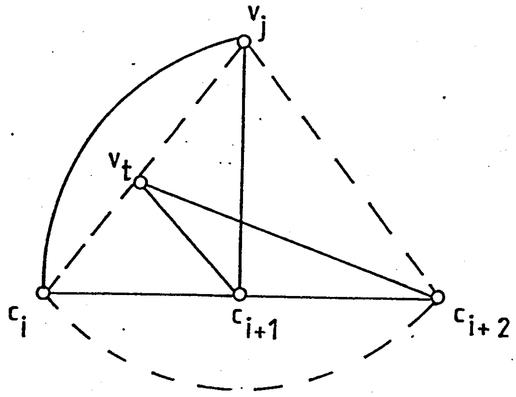


Figure 5.3

Similarly, if $p > j$, then v_p cannot be adjacent to any v_t for $t < j$. It follows that v_j is a separating vertex for the subgraph of G induced by Nc_{i+1}, which therefore cannot be Hamiltonian. This contradiction proves that c_i is adjacent to c_{i+2}. □

From Lemma 5.3 follow two corollaries.

Corollary 5.1

Let G, R, R' and C be as in Lemma 5.3, with c_{i+1} incident to the k-face in R but not in R'. Then $c_i c_{i+1} c_{i+2} c_i$ is a separating triangle for G.

Proof

This follows from the fact that c_i is adjacent to c_{i+2} and $\rho(c_{i+1}) > 2$. □

Corollary 5.2

Let G, R, R' and C be as in Lemma 5.3, with c_{i+1} incident to the k-face in R but not in R'. Then in any k-representation of G, c_i and c_{i+2} are incident to the k-face.

Proof

From Lemma 5.3(ii), c_i is adjacent to c_{i+2}. If we assume that c_i is not incident to the k-face for some k-representation of G, then

c_{i-1} is adjacent to c_{i+1}. Therefore the edge $c_{i-1}c_{i+1}$ is a chord of C which is skew with the chord c_ic_{i+2}. Therefore C can never be the boundary of a face of G (Theorems 2.5.1 and 2.5.3 of [O1]). But this is a contradiction, so that c_i must always be incident to the k-face in any k-represenration of G. The same can similarly be said for c_{i+2}. □

We thus see that if in a k-representation R, there exists a vertex c_{i+1} incident to the k-face, and which can be replaced by another vertex in some other k-representation, then $T = c_ic_{i+1}c_{i+2}c_i$ is a separating triangle. Hence if we define H_T as we did above when defining arches and spans, we obtain that H_T is an arch for C, which, when transferred to IntC (or to ExtC if the k-face of R is the unbounded face), gives another k-representation with c_{i+1} replaced by the replacement vertex y. The next lemma effectively tells us that y is the only vertex which can replace c_{i+1} on the k-face.

<u>Lemma 5.4</u>

Let R be a k-representation of a graph G, with the circuit C bounding the k-face in R, and labelled in the usual cyclic order $c_0c_1c_2\ldots c_{k-1}c_0$, and let $T = c_ic_{i+1}c_{i+2}c_i$ be a separating triangle of G. Let y be the replacement vertex of the span with VT as primary vertices. Then in any k-representation of G, either y or c_{i+1} must be incident to the k-face.

<u>Proof</u>

Let the components of $G - T$ be H_T and K_T, where $\overline{H}_T$ is the span with VT as primary vertices. Then $\overline{H}_T$ is a maximal planar graph in which y is incident to the face bounded by the circuit $c_iyc_{i+2}c_i$ (see Figure 5.4). However, K_T has some vertex adjacent to c_i and some vertex adjacent to c_{i+2}. Therefore there exists no plane representation of G in which both $c_ic_{i+1}c_{i+2}c_i$ and $c_iyc_{i+2}c_i$ are

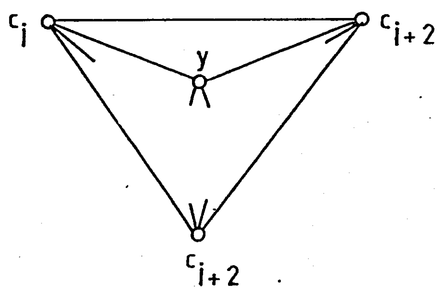


Figure 5.4

boundary circuits of faces.

Now, let us assume that the lemma is false. Then there exists a k-representation R' in which both y and c_{i+1} are incident solely to 3-faces. Now, by the above, in R', either the circuit $c_ic_{i+1}c_{i+2}c_i$ or else the circuit $c_iyc_{i+2}c_i$ is not the boundary of a face. Without loss of generality we may assume that $c_ic_{i+1}c_{i+2}c_i$ is not the boundary of a face in R'.

Now, we require that H, the subgraph of G induced by Nc_{i+1}, is Hamiltonian, since c_{i+1} is incident solely to 3-faces in R'. But since none of these faces is bounded by $c_ic_{i+1}c_{i+2}$, then $H - c_ic_{i+2}$ must also be Hamiltonian. But this leads to a contradiction as in the proof of Lemma 5.3. Therefore Lemma 5.4 is proved. □

We can now prove the central result of this section.

Theorem 5.2

Let R and R' be two k-representations of a graph G. Then R' can be changed into a representation equivalent to R by a sequence of arch transfers.

Proof

Let the vertices of the circuit C bounding the k-face in R be labelled in the usual cyclic order as $c_0, c_1, \ldots, c_{k-1}$. If C also bounds the k-face in R', then by Lemma 5.2, R is equivalent to R'. Thus we may assume that C does not bound the k-face in R'.

Let c_{i+1} be a vertex of C which is not incident to the k-face in R'. Therefore by Lemma 5.4, the replacement vertex of the span with c_i, c_{i+1}, c_{i+2} as primary vertices and c_{i+1} replaced vertex, must be incident to the k-face in R'. But then by the transfer in R' of the arch with c_i, c_{i+1}, c_{i+2} as vertices of attachment with C, we obtain a k-representation in which c_{i+1} is incident to the k-face and $c_i y c_{i+2} c_i$ bounds a face. We can repeat this process for every vertex c_j not incident to the k-face in R', and after a sequence of arch transfers we obtain a k-representation R'' in which all the vertices of C are incident to the k-face. Hence, by Lemma 5.1, the circuit C bounds the k-face in R''; therefore by Lemma 5.2, R'' is equivalent to R, and the theorem is proved. □

SECTION 5.2 - COLLAPSIBLE GRAPHS

In this section, unless otherwise stated, G will be a maximal planar graph of order at least 5 and with at least one vertex of valency 3. Let v be an ordinary vertex of G. Then G_v is ρv-representable. That ρv-representation of G_v having the neighbours of v incident to the ρv-face will be denoted by $R(v)$. By Lemma 5.1, the neighbours of v appear on the boundary of the ρv-face of $R(v)$ in a unique cyclic order. Thus we can label the set of neighbours of v in a cyclic order as $v_1, v_2, \ldots, v_{\rho v}$, and this labelling is unique up to choice of initial vertex and orientation. In this chapter, given any ordinary vertex v of G, by a labelling of the set of neighbours of v we shall mean such a cyclic labelling.

Now again, for an ordinary vertex v, consider G_v and $R(v)$. Let S be a span in $R(v)$, having primary vertices v_i, v_{i+1}, $v_{i+2} \in Nv$. We then say that the span S is incident to v. We shall now distinguish between certain types of span.

Let S be a maximal planar graph having the vertices of a boundary circuit of a face labelled v_i, v_{i+1}, v_{i+2}. Let G be a maximal planar graph having an ordinary vertex v incident to a span $S(v_i v_{i+1} v_{i+2})$ isomorphic to S (by an isomorphism which preserves labelling) and let y be the replacement vertex of this span. The ρv-representation of G_v obtained from $R(v)$ by replacing v_{i+1} by y on the boundary of the ρv-face is denoted by $R'(v)$. If for all such G, R_v is equivalent to $R'(v)$, we then say that $S(v_i v_{i+1} v_{i+2})$ is a symmetric span. A span which is not symmetric will be called asymmetric. Thus, for example, the spans $S(v_1 v_2 v_3)$ in Figure 5.5 are symmetric.

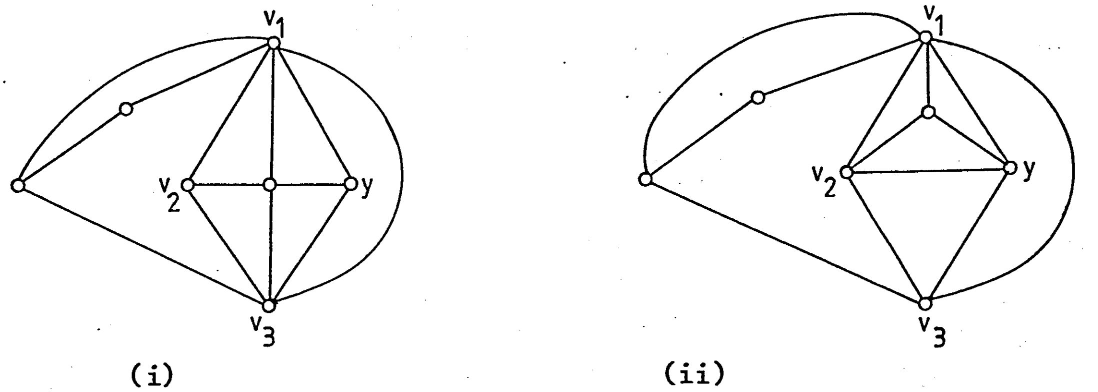
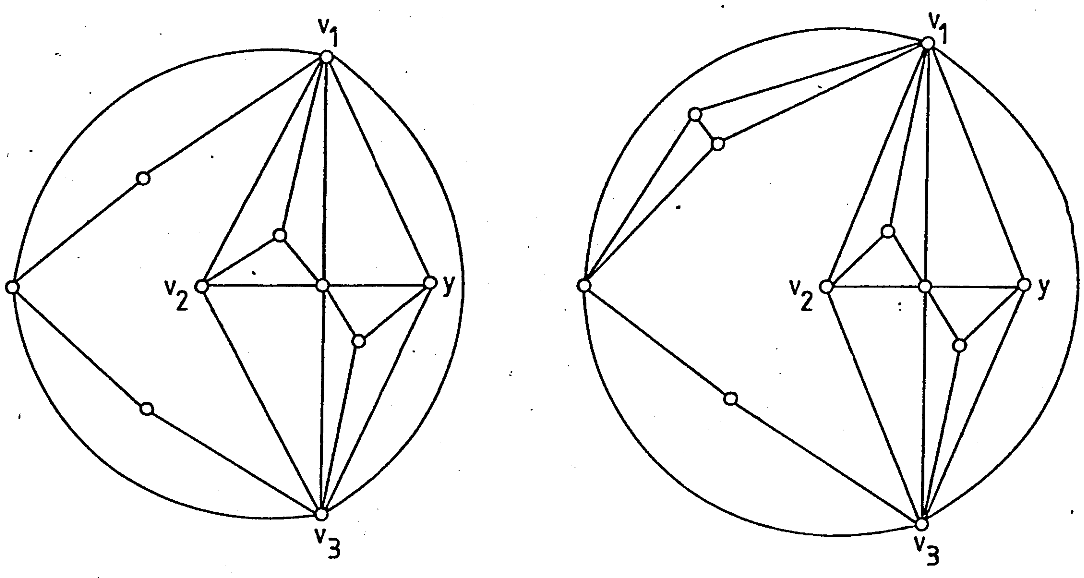


Figure 5.5

However, the span $S(v_1 v_2 v_3)$ of Figure 5.6 is asymmetric, because although in Figure 5.6(i) the interchange of v_2 and y gives equivalent 6-representations, this is not so in Figure 5.6(ii).

It is easy to see that the span $S(v_i v_{i+1} v_{i+2})$ is symmetric if and only if there is an automorphism ψ on S (considered as an unlabelled graph) such that $\psi v_{i+1} = y$, the replacement vertex, $\psi v_i = v_i$ and $\psi v_{i+2} = v_{i+2}$.



(i) (ii)

Figure 5.6

We note that if G is collapsible, then by Theorem 5.2, any ordinary vertex of G must be incident to at least one asymmetric span. An example of a collapsible graph is given in Figure 5.7. We also note that if a span is asymmetric, then it must have at least six vertices because the maximal planar graphs on four and five vertices are symmetric spans no matter which face is labelled.

We now have four results about spans, the first two following from the definitions.

Lemma 5.5

Let G be a maximal planar graph, v an ordinary vertex of G, and let $S(v_i v_{i+1} v_{i+2})$ be a span incident to v. Then $vv_i v_{i+1} v$ and $vv_{i+2} v_{i+1} v$ are boundary circuits of faces in G. □

The following lemma is a partial converse of the above.

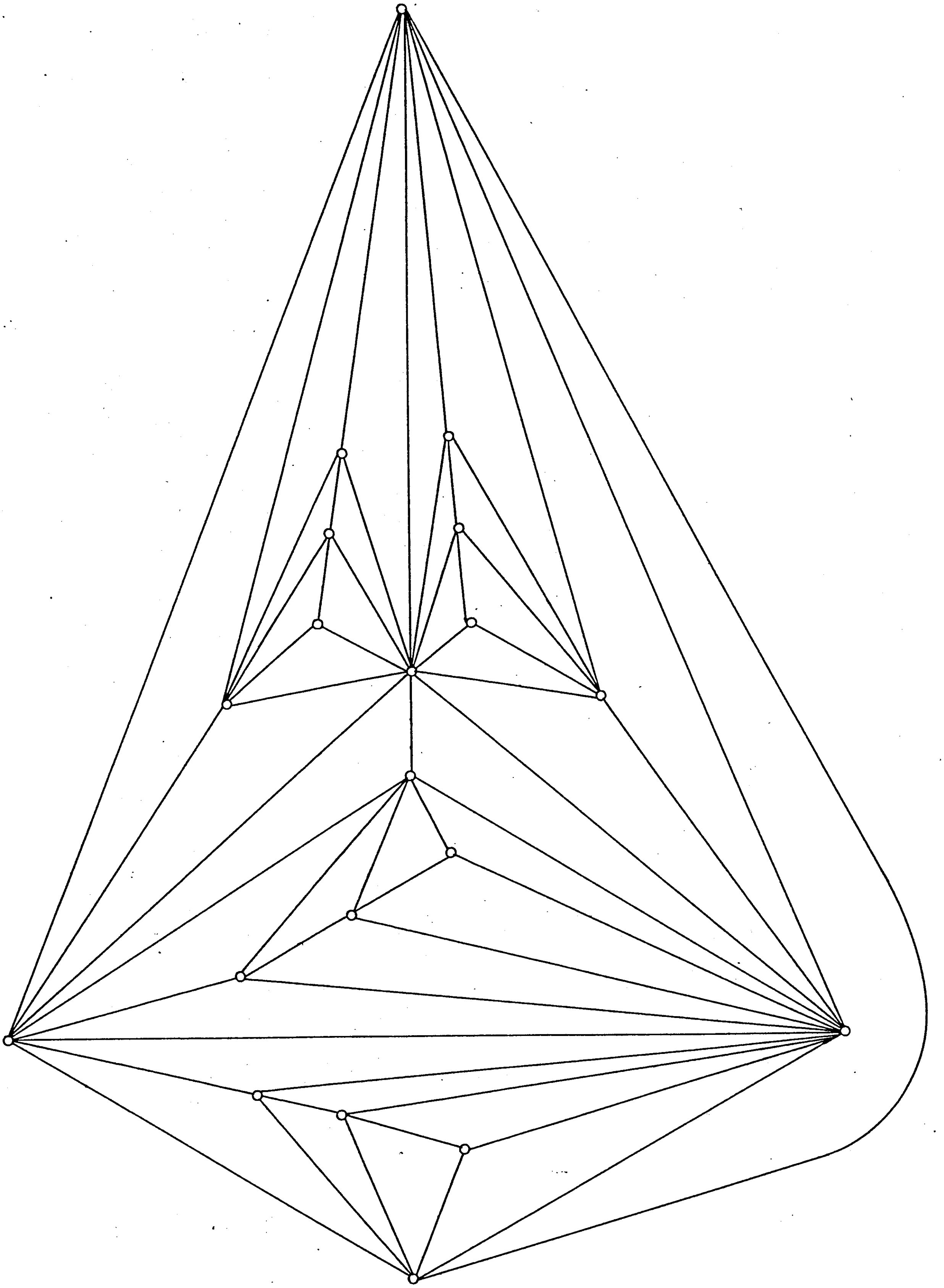


Figure 5.7

Lemma 5.6

Let G be a maximal planar graph, v an ordinary vertex of G, and x, y, z neighbours of v such that $vxyv$ and $vzyv$ are boundary circuits of faces in G. If $xyzx$ is a separating triangle for G, then x, y and z are the primary vertices of a span incident to v, with x and z as pivots. □

Lemma 5.7

Let G be a maximal planar graph, let v be an ordinary vertex of G, and let $abca$ be a triangle of G, with v adjacent to a, b, c. If v is incident to a span containing triangle $abca$, then this span must have a, b and c as primary vertices.

Proof

Let $S = S(xyz)$ be the span incident to v and containing triangle $abca$. Then x, y, z are the only vertices of S adjacent to v. But a, b, c are three vertices of S adjacent to v, so that triangle $abca$ is triangle $xyzx$ in some order. □

Lemma 5.8

Let G be a maximal planar graph, and let $S = S(w_1w_2w_3)$ be a span incident to an ordinary vertex w, with y as replacement vertex. If S is symmetric, then $\rho w_2 = 1 + \rho y$.

Proof

As we said above, since S is symmetric, then there is an isomorphism on S (considered as an unlabelled graph) which maps w_2 onto y. Therefore $\rho_S w_2 = \rho_S y$, and hence in G, $\rho w_2 = 1 + \rho y$. □

Before proceeding we require the following definition. Let K be a maximal planar graph, $|VK| > 4$, and let $abca$ be a face of K. Let K have the property that for any ordinary vertex $v \in VK - \{a,b,c\}$, either (i) v is incident to a span containing triangle abca,

or (ii) v is adjacent to a, b and c, and two of $vabv$, $vacv$, $vcbv$ are boundary circuits of faces.

Then we say that K envelopes triangle abca.

Lemma 5.9

Let abca be a face of a maximal planar graph K, and let K envelope triangle abca. Then there exists an ordinary vertex in K adjacent to a, b and c.

Proof

We assume that there is no vertex in K adjacent to a, b and c. Now, since the order of K is greater than 4, then there exists at least one ordinary vertex in $VK - \{a,b,c\}$, and every such vertex must be incident to a span containing triangle abca. Among all such spans, let $S = S(w_1w_2w_3)$, incident to the ordinary vertex w, be minimal, in the sense that no other span containing abca can have less vertices than S. From Lemma 5.5, it follows that ww_1w_2w and ww_3w_2w are boundary circuits of faces (see Figure 5.8)

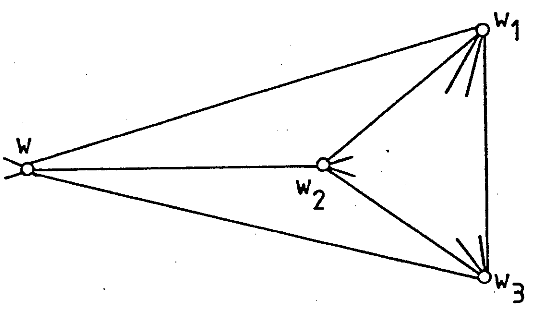


Figure 5.8

We define $(w,w_1,w_3)_{in}$ as that component of $K - \{w,w_1,w_3\}$ which contains w_2, whereas $(w,w_1,w_3)_{out}$ is defined as the other component of $K - \{w,w_1,w_3\}$. Similarly we define $(w_1,w_2,w_3)_{out}$ as that component of $K - \{w_1,w_2,w_3\}$ containing w, whereas $(w_1,w_2,w_3)_{in}$ is the other component of $K - \{w_1,w_2,w_3\}$. Thus it follows that triangle abca $\subset \overline{(w_1,w_2,w_3)}_{in} \subset \overline{(w,w_1,w_3)}_{in}$.

Now, at least one of w_1, w_2 or w_3 must be different from a, b and c. We thus consider three cases.

Case 1 $w_2 \notin \{a,b,c\}$

We first note that since w_2 is a primary vertex of a span, then w_2 is an ordinary vertex. But we are assuming that no vertex of K is adjacent to a, b and c. Thus the fact that K envelopes triangle abca implies that w_2 is incident to a span containing abca. Let x, y, z be the primary vertices of this span.

First we note that none of x, y or z can be in $(w,w_1,w_3)_{out}$, since K is planar. Also, $\{x,y,z\}$ cannot be $\{w,w_1,w_3\}$, because the span incident to w_2 with w, w_1 and w_3 as primary vertices is $\overline{(w,w_1,w_3)}_{out}$, which does not contain triangle abca.

Thus, none of x, y, z can be w, because if x, say, is w, then one of y or z, say y, is in $(w_1,w_2,w_3)_{in}$, and therefore w is adjacent to y, making K nonplanar. Thus we have that triangle xyzx is a subgraph of $\overline{(w_1,w_2,w_3)}_{in}$.

Now, define $(x,y,z)_{in}$ to be that component of K - $\{x,y,z\}$ not containing w_2, and $(x,y,z)_{out}$ as that component of K - $\{x,y,z\}$ containing w_2. Thus tha span incident to w_2 with x, y and z as primary vertices is either $\overline{(x,y,z)}_{in}$ or $\overline{(x,y,z)}_{out}$. But the minimality of $S(w_1w_2w_3)$ implies that it cannot be $\overline{(x,y,z)}_{in}$. Thus $\overline{(x,y,z)}_{out}$ is a span incident to w_2 and which contains w_2, a contradiction.

Case 2 $w_3 \notin \{a,b,c\}$

This is similar to Case 3 below.

<u>Case 3</u> <u>$w_1 \notin \{a,b,c\}$</u>

Again w_1 is ordinary, and hence is incident to a span containing triangle abca, and again we let x, y, z be the primary vertices of this span. We cannot have that xyzx is ww_2w_3w in any order, since the latter is a boundary circuit of a face. Thus either triangle xyzx is a subgraph of $(\overline{w,w_1,w_3})_{out}$ or else it is a subgraph of $(\overline{w_1,w_2,w_3})_{in}$.

<u>Case 3.1</u> <u>Triangle $xyzx \subset (\overline{w,w_1,w_3})_{out}$</u>

Let $(x,y,z)_{in}$ be that component of $K - \{x,y,z\}$ not containing w_1, and let $(x,y,z)_{out}$ be the other component. Thus $abca \subset (x,y,z)_{out}$. Therefore the span incident to w_1 with x, y, z as primary vertices must be $(\overline{x,y,z})_{out}$. However this is impossible since $(\overline{x,y,z})_{out}$ contains w_1.

<u>Case 3.2</u> <u>Triangle $xyzx \subset (\overline{w_1,w_2,w_3})_{out}$</u>

We define $(x,y,z)_{in}$ as that component of $K - \{x,y,z\}$ not containing w_1 while $(x,y,z)_{out}$ is the other component. The minimality of $S(w_1w_2w_3)$ implies that the span incident to w_1 with x, y, z as primary vertices is $(\overline{x,y,z})_{out}$. However this is impossible, since $(\overline{x,y,z})_{out}$ contains w_1.

We have thus obtained a contradiction in every possible case. Thus there exists a vertex v adjacent to a, b and c in K. However, since the order of K exceeds 4, and since abca is a boundary circuit of a face, it then follows that v is an ordinary vertex. □

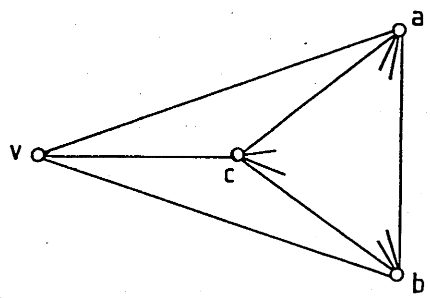
<u>Lemma 5.10</u>

Let G be a maximal planar graph, and let H be a maximal planar subgraph of G such that $\langle C(G,H)\rangle$ is the 3-circuit vaba. Let there be a vertex c in H such that vcav and vcbv are faces in H. Let w be an ordinary vertex in G, $w \in VG - VH$, and let

$S = S(w_1 w_2 w_3)$ be a span incident to w and containing triangle abca. Then S contains triangle vabv.

<u>Proof</u>

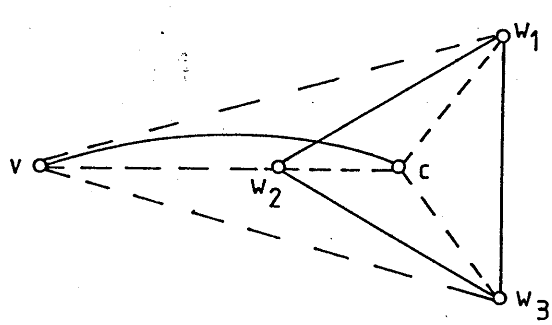
We can represent H as in Figure 5.9, where abca may or may not bound a face.



<u>Figure 5.9</u>

Let $T = w_1 w_2 w_3 w_1$. We define T_{out} as that component of $G - VT$ which contains w, T_{in} being the other component. Therefore $S = \overline{T}_{in}$, since S cannot contain w. Now, c cannot be a vertex of T, since w is adjacent to the three vertices of T but not to c. But abca is in S, so that c is in T_{in}.

We assume that S does not contain vabv, so that v is in T_{out}. But both $\overline{T}_{in}$ and $\overline{T}_{out}$ are maximal planar graphs, and hence 3-connected. Therefore the graph in Figure 5.10, a subdivision of K_5, is a subgraph of G, a contradiction.



<u>Figure 5.10</u>

Thus S must contain triangle vabv. $\square$

Before we can state the principal theorems of this section, we require the definition of a special type of maximal planar graph, and some related results.

Let G be a maximal planar graph of order ν and for which there exists a sequence of nested subgraphs $G_1, G_2, \ldots, G_{\nu-3}$ satisfying the following conditions:

(i) $G = G_1$ and $G_{\nu-3} = K_3$;

(ii) each G_i $(i = 1,2,\ldots,\nu-5)$ has exactly two vertices of valency 3;

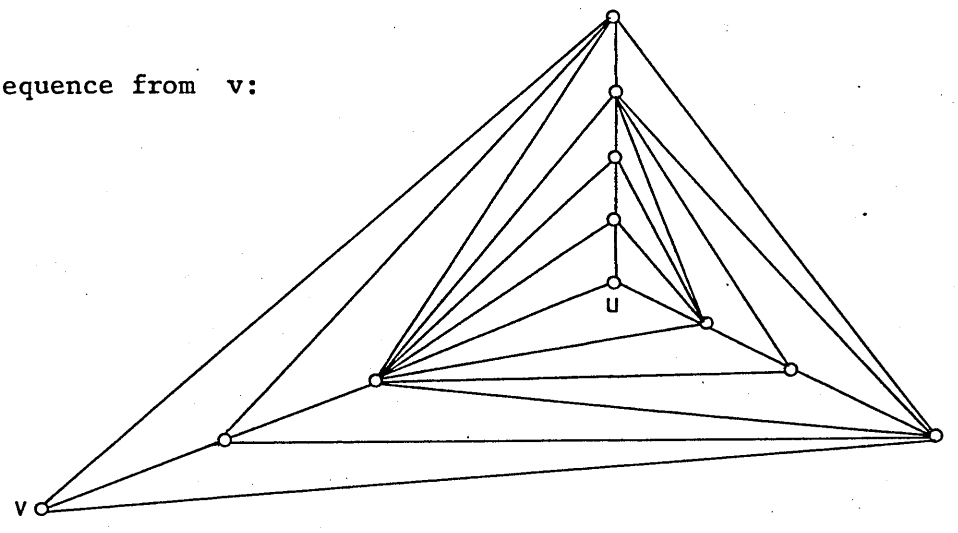
(iii) G_{i+1} is obtained from G_i by deleting a vertex of valency 3.

We shall call such a graph a <u>stitching graph</u>.

Any maximal planar graph in <u>normal form</u> (as defined in [01, p.9]) is a stitching graph. Another example of a stitching graph is given by the graph of Figure 5.11.

Stitching sequence from u:
(2,2,1,3)

Stitching sequence from v:
(1,1,2,4)



<u>Figure 5.11</u>

The process (described in the above definition) of reducing a stitching graph G, with two vertices u and v of valency 3, to a triangle, but carried out such that we first delete the vertex u, and then, at each subsequent step we delete the *new* vertex of valency 3 created by the deletion of the previous 3-vertex will be called the <u>unstitching of G from u to v</u>. The unstitching from v to u is similarly defined. Note that in the unstitching of G, from u say, we obtain K_4 immediately before the deletion of the last vertex. At that stage we can delete any vertex different from v to arrive at the final triangle.

Let us assume that at a stage in the unstitching of G, x is the next vertex to be deleted, and y the new 3-vertex created by the deletion of x. Let a and b be the neighbours of x, apart from y, when x is to be deleted. Then a and b will be called the extra neighbours of x. If y in its turn has the same extra neighbours as x, we then say that x and y are in the same row. Since the deletion of the last vertex results in a triangle, and since we have a choice of three vertices to delete at the last step, we shall always consider the unstitching to be carried out in such a way that the last three vertices which are deleted are in the same row. Thus if we define the length of a row to be the number of vertices in the row, we then have that the last row in the unstitching of G is always of length greater than 2.

The stitching sequence of G from u to v is defined as the sequence of lengths of the rows in the order in which they occur in the unstitching from u. The stitching sequence from v to u is similarly defined. Thus the stitching sequence from u to v for the graph in Figure 5.11 is (2,2,1,3) while the sequence from v to u is (1,1,2,4).

In Propositions 5.1 to 5.4 we list some properties of stitching sequences. The two vertices of valency 3 in G will be u and v, and the stitching sequences of G from u to v and from v to u will be respectively $(a_1,a_2,\ldots,a_p)$ and $(b_1,b_2,\ldots,b_q)$.

Proposition 5.1 $\sum_1^p a_i = \sum_1^q b_i = \nu - 3$, where $\nu = \nu G$. Also, $a_p \geq 3$ and $b_q \geq 3$, except when $\nu < 6$. If $\nu < 6$, then G is in normal form and therefore has only one row.

Proposition 5.2 If G has more than one row, then $b_1 = a_p - 2$ and $a_1 = b_q - 2$.

Proof

Since G has more than one row then G must have more than five vertices, so that $a_p \geq 3$. Thus in the unstitching of G from u, we have that the last row is as in Figure 5.12, where $r = a_p$.

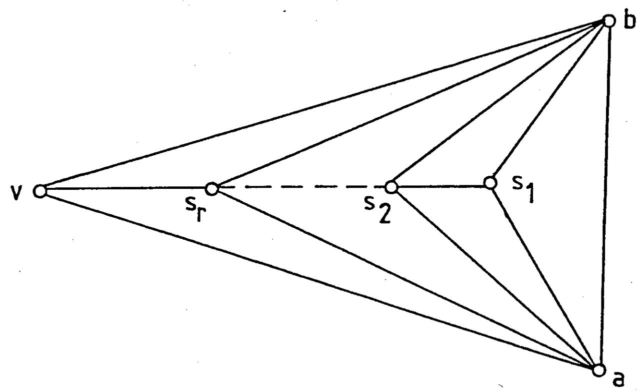


Figure 5.12

Therefore (referring to Figure 5.12) the vertex w, which was removed before s_1 was either in the interior of triangle as_1s_2a or else in the interior of triangle bs_1s_2b, since we are assuming that G has more than one row. We can with no loss of generality assume that w is in the interior of triangle bs_1s_2b, so that the rest of G (which is not shown in Figure 5.12) is also in the interior of bs_1s_2b.

Now let us consider the unstitching of G from v. Then the only vertices in the first row are v, s_r, $s_{r-1},\ldots,s_4$, so that $b_1 = a_p - 2$. Similarly we can show that $a_1 = b_q - 2$. □

Proposition 5.3 $p = q$.

Proof

We shall prove this by induction on the number of vertices of G. But towards this end we first have to consider what happens to the stitching sequences of G when one of the vertices u or v is deleted from G. Let us assume that u is deleted. We now have to find the stitching sequences for G_u. Let w be the new 3-vertex.

Case 1 $a_1 = 1$

Then the stitching sequence from w for G_u is $(a_2, a_3, \ldots, a_p)$. We now have to find the sequence for G_u starting from v.

Since $a_1 = 1$, then $b_q = 3$. Therefore before we delete u, the final row in the unstitching of G from v is as in Figure 5.13, where x is the next vertex to be deleted.

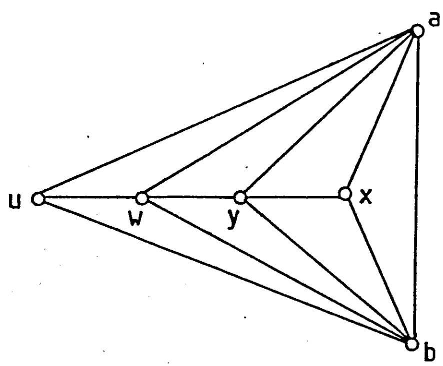


Figure 5.13

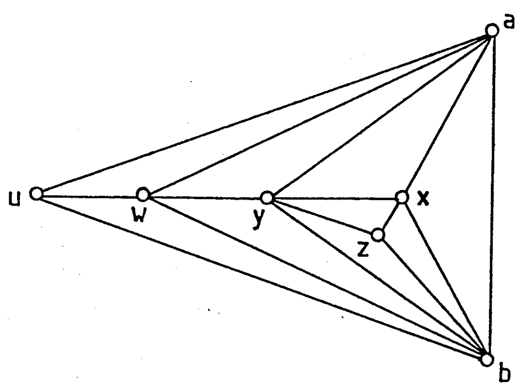


Figure 5.14

Since the last row had length 3, then the vertex z which was deleted before x must have been in the interior of bxyb or else in the interior of axya (referring to Figure 5.13). We can assume with no loss of generality that z was in Int(bxyb), so that, before deleting z, we had the graph shown in Figure 5.14.

Therefore in the final stages of the unstitching of G_u from v to w we have Figure 5.14 with u deleted, and we see that x and a are now in the same row as z. It follows that the stitching sequence of G_u from v to w is $(b_1, b_2, \ldots, b_{q-1} + 2)$.

Case 2 $a_1 > 1$

Thus when u is deleted, the stitching sequence in G_u starting from w is $(a_1 - 1, a_2, \ldots, a_p)$. We now consider the unstitching of G from v to u. Since $a_1 \geq 2$, then $b_q \geq 4$. Therefore in the unstitching of G from v to u, we have that the last row is as shown in Figure 5.15, where the final row has length greater than 3.

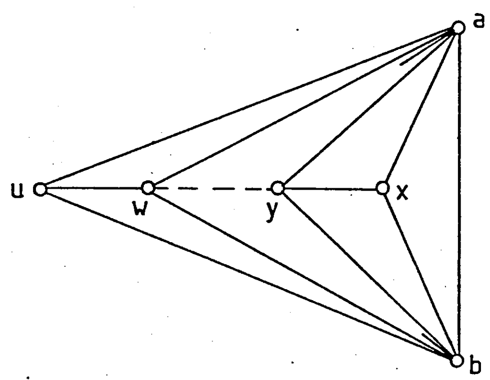


Figure 5.15

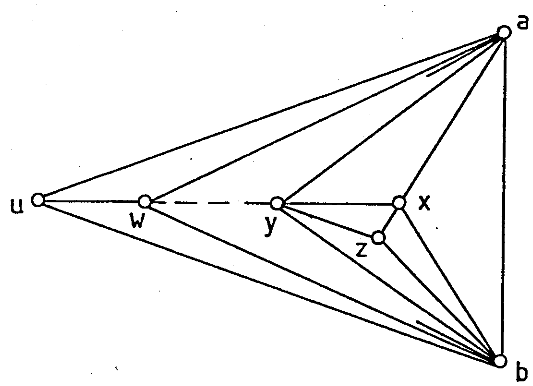


Figure 5.16

Again, if z were the vertex deleted before x, then we can assume that z is in the interior of triangle bxyb (referring to Figure 5.15). Therefore in the last stages of the unstitching of G_u from v to w we have Figure 5.16 with the vertex u deleted. Hence the vertices x,y,...,w are still in a row different from that of z. Thus the stitching sequence from v to w in G_u is $(b_1, b_2, \ldots, b_q - 1)$.

We can now prove that p = q. This is easily verified when ν, the order of G is 4 or 5. We therefore assume that the result is true for all graphs with less than ν vertices, and consider G_u. Then when $a_1 = 1$ (that is, as in Case 1 above), the stitching sequences in G_u are $(a_2, a_3, \ldots, a_p)$ and $(b_1, b_2, \ldots, b_{q-1} + 2)$. Therefore $p - 1 = q - 1$, by the induction hypothesis, and hence p = q. When $a_1 \geq 2$ (that is, as in Case 2 above), the stitching sequences in G_u are $(a_1 - 1, a_2, \ldots, a_p)$ and $(b_1, b_2, \ldots, b_q - 1)$, so that again we obtain that $p = q$. □

Proposition 5.4 If $i \in \{0,1,2,\ldots,p-3\}$, then $a_{i+2} = b_{p-1-i}$.

Proof

We delete u from G, thus creating a new vertex w_1 of valency 3. If $a_1 \geq 2$, then we obtain $(a_1 - 1, a_2, \ldots, a_p)$ as the stitching sequence in G_u from w_1 to v, and $(a_p - 2, b_2, \ldots, b_{p-1}, a_1 + 1)$ as the sequence from v to w_1.

We then delete w_1, obtaining w_2 as the new 3-vertex. If $a_1 - 1 \geq 2$, then we get the stitching sequences $(a_1 - 2, a_2, \ldots, a_p)$ from w_2, and $(a_p - 2, b_2, \ldots, b_{p-1}, a_1)$ from v.

We repeat this process until $a_1 - j = 1$, when we have the sequences $(1, a_2, \ldots, a_p)$ from w_j, and $(a_p - 2, b_2, \ldots, b_{p-1}, 3)$ from v. If we delete w_j, we then obtain the stitching sequences $(a_2, a_3, \ldots, a_p)$ and $(a_p - 2, b_2, \ldots, b_{p-1} + 2)$. Therefore by Proposition 5.2, we deduce that $b_{p-1} + 2 = a_2 + 2$, so that $b_{p-1} = a_2$. Similarly, by continuing the above process, we obtain that $a_{i+2} = b_{p-1-i}$, for $i = 0,1,\ldots,p-3$. □

The graph in Figure 5.11 illustrates Propositions 5.1 to 5.4.

We can now prove the principal results of this section.

Theorem 5.3

Let K be a maximal planar graph and let $abca$ be a boundary circuit of a face of K. If K envelopes triangle $abca$, then K is a stitching graph, and one of a, b or c has valency 3 in K.

Proof

By Lemma 5.9 there exists an ordinary vertex v_1 in K adjacent to a, b and c, and since K envelopes $abca$, then two of abv_1a, acv_1a, bcv_1b are boundary circuits of faces. We can assume with no loss of generality that acv_1a and bcv_1b are faces, so that c is a 3-vertex in K, proving one part of the theorem.

Now, if there exists no ordinary vertex of K in $VK - \{a,b,c,v_1\}$, then there is just one more vertex of K, adjacent to v_1, a and b, so that K would be a stitching graph. We may therefore assume that there exists at least one ordinary vertex of K in $VK - \{a,b,c,v_1\}$. But K envelopes triangle $abca$, so that any such ordinary vertex must be incident in K to a span containing $abca$. By Lemma 5.10, any such span must contain triangle abv_1a. Therefore the graph $K - c$ envelopes triangle abv_1a. Thus by the above, there exists a vertex $z \in \{v_1,a,b\}$ which has valency 3 in $K - c$.

We can now apply induction on the number ν of vertices of K. We first note that the theorem is certainly true for $\nu = 5$, since the only maximal planar graph on 5 vertices is a stitching graph. Thus we assume that the theorem is true for any graph with less than ν vertices, so that $K - c$ is a stitching graph. But z is a 3-vertex in $K - c$, and hence there exists another vertex w of valency 3 in $K - c$, and $w \notin \{v_1,a,b\}$. But K is obtained from $K - c$ by adding the 3-vertex c, and joining it to v_1, a and b. Therefore K is a stitching graph with c and w as the two vertices of valency 3. □

<u>Lemma 5.11</u>

Let G be a collapsible graph and let $S = S(w_1w_2w_3)$ be an asymmetric span incident to an ordinary vertex w in G. If S has minimal order among all asymmetric spans in G, then S must satisfy these three conditions:

(i) If u is an ordinary vertex in S, different from w_1, w_2, w_3, and S' is an asymmetric span incident to u in S, then S' must contain triangle $w_1w_2w_3w_1$;

(ii) S envelopes triangle $w_1w_2w_3w_1$;

(iii) if for any $j = 1, 2$ or 3, w_j is ordinary in S, then w_j

cannot be incident to an asymmetric span in S.

Proof

To prove (i) we first note that $|VS'| < |VS|$, so that the minimality of S implies that S' cannot be incident to u in G. Thus if S $S' = S'(pqr)$, this can arise either if a face of S' is no longer a face in G, or if one of the triangles $upqu$, $urqu$ no longer bounds a face in G. However, the only triangle which bounds a face in S but not in G is $w_1w_2w_3w_1$. Therefore the second of the above alternatives is impossible, because since u is different from w_1, w_2, w_3, then both triangles $upqu$, $urqu$ remain boundary circuits of faces in G. Thus we must have that a face of S' is no longer a face in G. Hence S' must contain triangle $w_1w_2w_3w_1$. This proves (i).

Now, let t be an ordinary vertex of S different from w_1, w_2, w_3. If t is incident to an asymmetric span in S, then by (i), t is incident to a span in S which contains $w_1w_2w_3w_1$. Thus to prove (ii) we can assume that in S, t is incident to no asymmetric span. However, G is collapsible, so that in G, t must be incident to an asymmetric span. We assume that no span incident to t in S contains triangle $w_1w_2w_3w_1$. Thus any such span remains the same in S and in G. Hence in G, t must be incident to a span to which it is not incident in S, and this cannot arise by the creation of new neighbours of t, since $t \notin \{w_1,w_2,w_3\}$. Therefore there exist vertices x, y, z neighbours of t, which are primary vertices of a span incident to t in G but not in S. Hence by Lemmas 5.5 and 5.6, $xyzx$ is a separating triangle in G but not in S. It follows that triangle $xyzx$ is $w_1w_2w_3w_1$ in some order, so that t is adjacent to w_1, w_2 and w_3. But now, in G, t is incident to a span with w_1, w_2, w_3 as primary vertices. Therefore by Lemma 5.6, two of tw_1w_2t, tw_1w_3t, tw_2w_3t are boundary circuits of faces in G, and hence in S. It follows that S envelopes triangle $w_1w_2w_3w_1$.

We shall now prove (iii) for $j = 1$. Thus let w_1 be ordinary in S, and let $S'' = S''(abc)$ be an asymmetric span incident to w_1 in S. Then $|VS''| < |VS|$. Therefore the minimality of S implies that S'' cannot be incident to w_1 in G. This can arise either if a face of S'' is no longer a face in G, or if one of the triangles w_1abw_1, w_1cbw_1 no longer bounds a face in G. However the first of these alternatives is impossible, because since S'' is a span incident to w_1, then it cannot contain triangle $w_1w_2w_3w_1$. Therefore one of w_1abw_1 or w_1cbw_1, say w_1abw_1, does not bound a face in G. Therefore w_1abw_1 is triangle $w_1w_2w_3w_1$ in some order. This means that w_2 and w_3 are primary vertices of S'', and hence are ordinary in S. But S envelopes triangle $w_1w_2w_3w_1$, so that we have a contradiction to Theorem 5.3. Therefore w_1 cannot be incident to an asymmetric span in S. □

Theorem 5.4

Let G be a collapsible graph. Then there exists an ordinary vertex u in G, such that u is incident to an asymmetric span S, with a, b, c as primary vertices, and such that S is one of the eight types of graph shown in Figure 5.17, σ being a permutation of $\{a,b,c\}$.

Proof

Since G is collapsible, then every ordinary vertex in G is incident to an asymmetric span. Let S be an asymmetric span incident to an ordinary vertex u in G, such that S has minimal order among all asymmetric spans of G, and let a, b, c be the primary vertices of S. We now claim that S is one of the eight types of graph shown in Figure 5.17. (The labelling of the other vertices will be required in the next theorem.)

To prove this we first note that S must satisfy conditions (i),

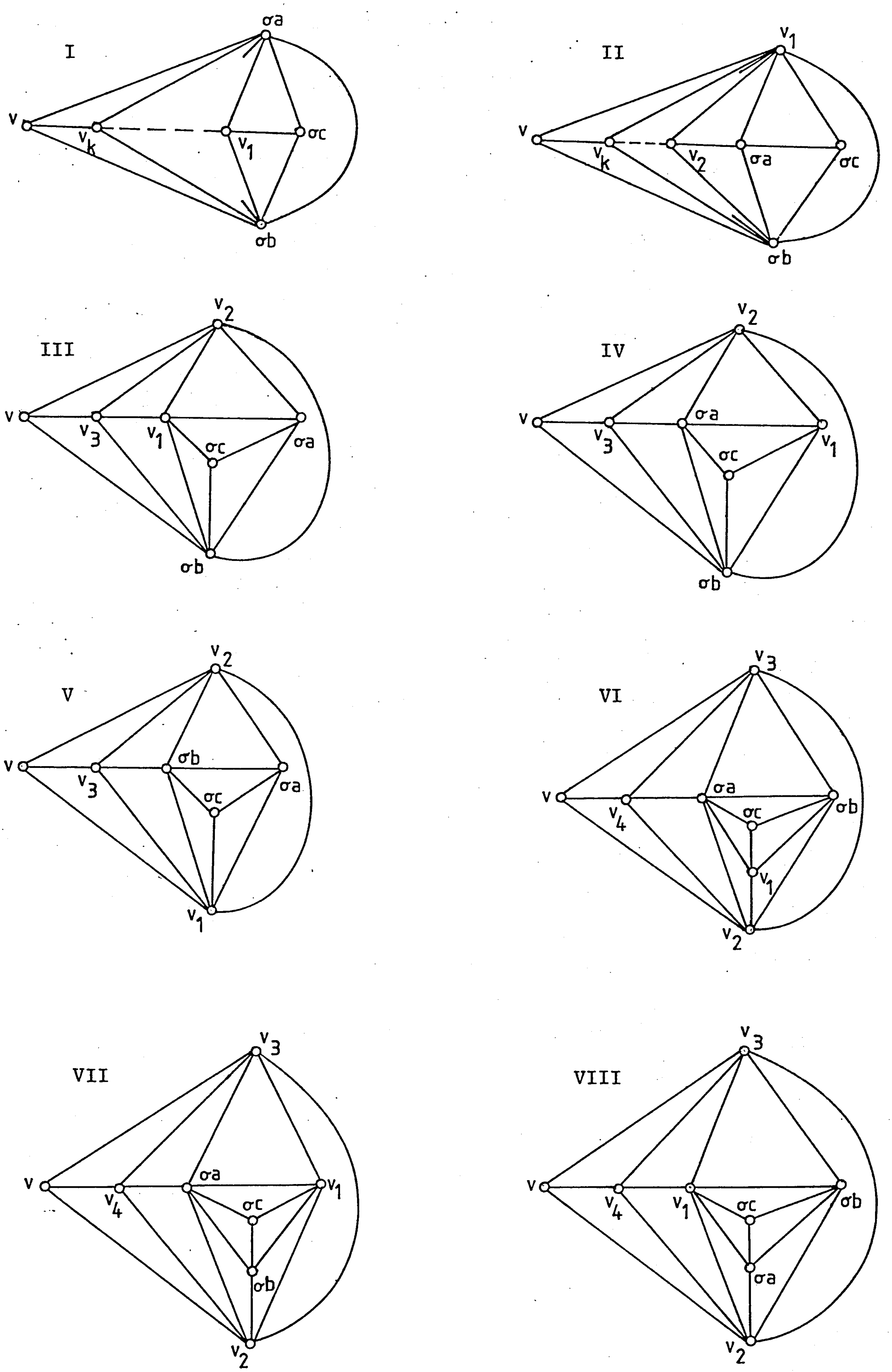

I
σa
v
v_k
v_1
σc
σb
II
v_1
v
v_k
v_2
σa
σc
σb
III
v_2
v
v_3
v_1
σc
σa
σb
IV
v_2
v
σa
v_3
σc
v_1
σb
V
v_2
v
σb
v_3
σc
σa
v_1
VI
v_3
v
σa
v_4
σc
σb
v_1
v_2
VII
v_3
v
σa
v_4
σc
v_1
σb
v_2
VIII
v_3
v
v_4
v_1
σc
σb
σa
v_2

Figure 5.17

(ii) and (iii) of Lemma 5.11. Therefore by (ii) and by Theorem 5.3, S must be a stitching graph and one of a, b, c has valency 3 in S. We can, with no loss of generality, assume that c has valency 3 in S. Let v be the other 3-vertex in S, and let $(a_1, a_2, \ldots, a_p)$ be the stitching sequence of S from c to v.

We first note that if S has only one row, then it is in normal form. In this case, since c has valency 3, we obtain that S is a graph of type I or II. We can therefore assume that S has more than one row, that is, $p > 1$.

We first show that $a_p = 3$. Let us assume on the contrary that $a_p > 3$. Then since S has more than one row, it is as in Figure 5.18, where the rest of S is in the interior of triangle xyzx, and where xyrx, zyrz are boundary circuits of faces. Therefore $\rho r \geq 6$ and $\rho t = 4$. It follows by Lemma 5.8, that the span S(xrz) incident to y is asymmetric. However this span does not contain triangle abca, so that we have a contradiction to Lemma 5.11. Therefore $a_p = 3$.

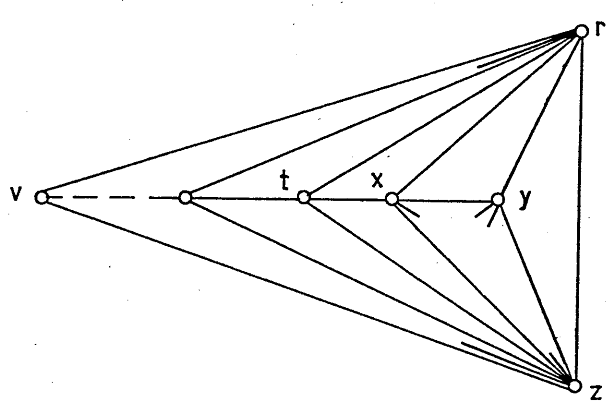


Figure 5.18

We now show that $a_{p-1} = 1$. Let us assume on the contrary that a_{p-1} is greater than 1. Then S is as in Figure 5.19, where xwyx and zwyz bound faces. Therefore $\rho y = 4 = \rho t$. Hence by Lemma 5.8, the span S(xyz) incident to w, with replacement vertex t, is not symmetric. But this span does not contain abca, and hence we obtain

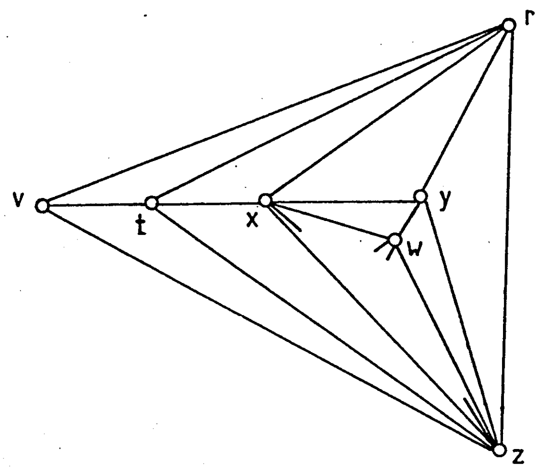


Figure 5.19

a contradiction to Lemma 5.11. Therefore $a_{p-1} = 1$.

Now, we note that if S has only two rows, then it is as shown in Figure 5.20. But then, since c has valency 3, it must be vertex w. Therefore one of the triangles xwyx, xwzx, ywzy must be triangle abca. It follows that S is one of types III, IV, V. We can therefore assume that S has more than two rows.

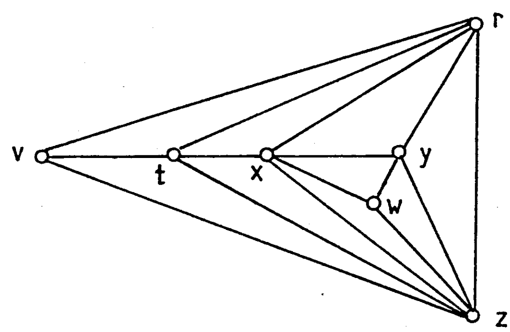
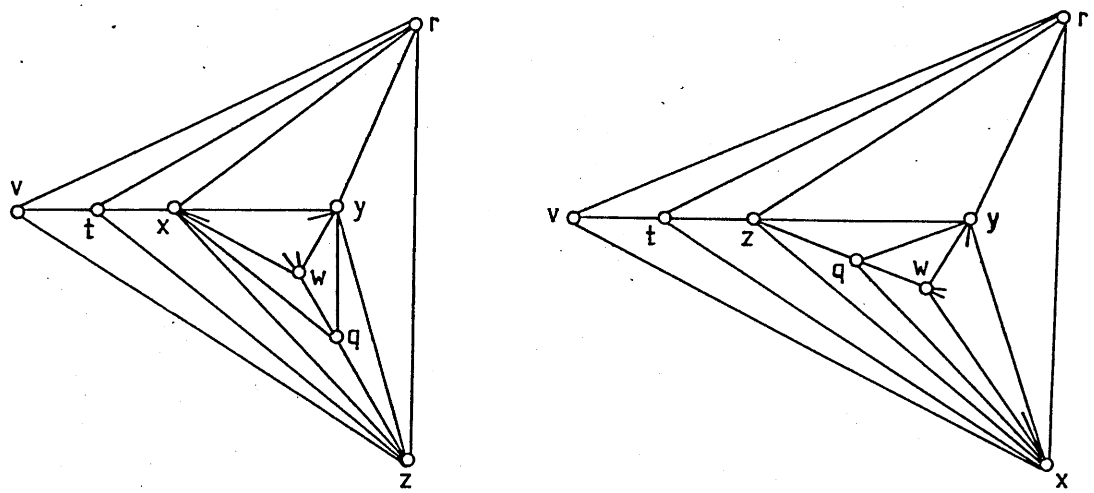


Figure 5.20

We next show that $a_{p-2} = 1$. If on the contrary we assume that a_{p-2} is greater than 1, then S is as in Figure 5.21(i) or as in Figure 5.21(ii), where in both cases xqwx and yqwy bound faces.

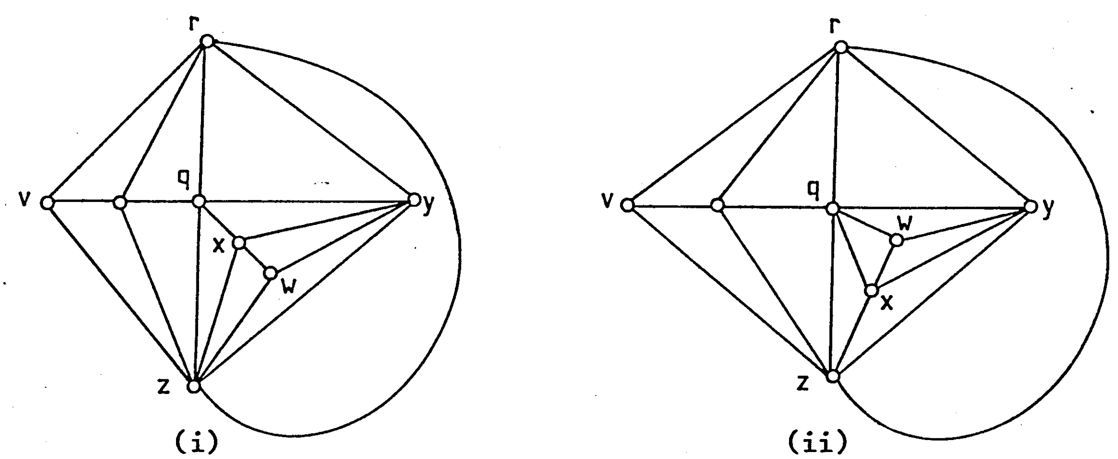


(i) (ii)

Figure 5.21

But then in either case, the span S(yqx) incident to w, and with replacement vertex r, is asymmetric by Lemma 5.8, since $\rho q = 4$ and $\rho r = 5$. But this span does not contain triangle abca, so that we again have a contradiction to Lemma 5.11. Therefore $a_{p-2} = 1$.

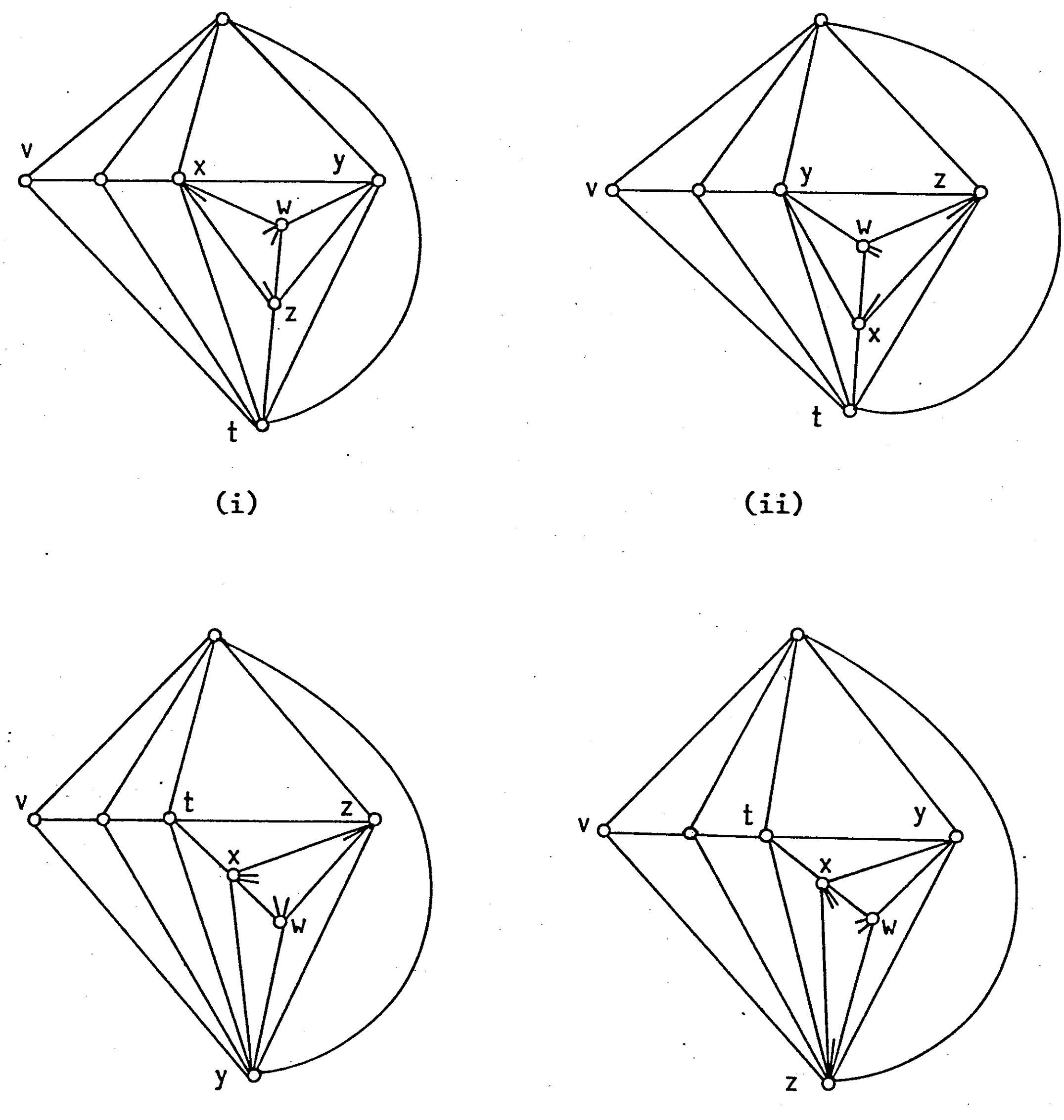
Now, if S has only three rows, then it is as in Figure 5.22(i) or as in Figure 5.22(ii). But in Figure 5.22(i) the span S(yqz) incident to x is asymmetric and does not contain abca. Therefore S must be as in Figure 5.22(ii). Hence triangle abca must be one of triangles qwyq, qwxq, ywxy, giving types VI, VII, VIII respectively.



(i) (ii)

Figure 5.22

The proof will now be complete if we can show that S cannot have more than three rows. Thus we assume for contradiction that S has more than three rows. Then it must be as one of the graphs in Figure 5.23, where in each case the rest of S is in the interior of triangle wxzw.



(i) (ii) (iii) (iv)

Figure 5.23

But in each of the graphs in Figure 5.23 we have that $\rho y \neq \rho t + 1$. Therefore by Lemma 5.8 the span S(xyz) incident to w with t as replacement vertex is asymmetric. But this span does not contain triangle abca, so that we have a contradiction to Lemma 5.11. Hence S cannot have more than three rows. □

Before stating Theorem 5.5, which is the principal result of this section, we have one final definition. Let G be a maximal planar graph, and let w be an ordinary vertex of G with the property that w is incident to only one asymmetric span $S(w_1w_2w_3)$ having replacement vertex y. Then if $\rho w_2 \neq \rho y + 1$ we say that w is a <u>good vertex</u>.

<u>Theorem 5.5</u>

Every collapsible graph has a good vertex.

<u>Proof</u>

Let G be a collapsible graph. Then by Theorem 5.4 there exists an ordinary vertex u incident to an asymmetric span S, where S is one of the eight types of graph in Figure 5.17.

We now consider these eight cases. We let a, b, c be the primary vertices of S, and we consider the different cases which arise, depending on which one of a, b, c is the replaced vertex of S considered as a span incident to u. We take σ in Figure 5.17 to be the identity permutation. (For easy verification, the proof of Theorem 5.5 is summarized in a table which we give below.)

<u>Case I</u> <u>S is of type I</u>

It is clear that since S is asymmetric, then c cannot be the replaced vertex of S. We can therefore assume, with no loss of generality, that b is the replaced vertex of S, so that $k \geq 2$, since S is asymmetric.

Now, we have that $ubau$ and $ubcu$ bound faces in G. Therefore v_k is incident to only one asymmetric span $S(bv_{k-1}a)$, and this has replacement vertex u. But $\rho u \geq 4 = \rho v_{k-1}$, therefore v_k is a good vertex of G.

Case II S is of type II

Then the pivots of S cannot be a and c, as otherwise S would be symmetric. We therefore have two cases to consider.

II.1 The pivot vertices of S are a and b

We first note that $k \geq 2$, as otherwise S would be symmetric. We therefore have:

II.1.1 $k \geq 3$

Then v_k is not adjacent to s, so that v_k is adjacent to v_1, v, b and v_{k-1}. Moreover, $ucau$ and $ucbu$ bound faces in G. Therefore the only asymmetric span incident to v_k is $S(v_1v_{k-1}b)$, with c as replacement vertex. However, $\rho v_{k-1} = \rho c = 4$. Therefore v_k is a good vertex.

II.1.2 $k = 2$

Therefore the only asymmetric span incident to v_2 is $S(v_1ab)$, with c as replacement vertex. But $\rho c = 4$ and $\rho a > 5$. Therefore v_2 is a good vertex.

II.2 The pivot vertices of S are b and c

Therefore $uabu$ and $uacu$ bound faces in G. Moreover, $k \geq 2$, as otherwise S would be symmetric. We therefore have:

II.2.1 $k \geq 3$

Then v_k is not adjacent to a, and hence v_k is adjacent to v_1, v, b, v_{k-1} in G. It follows that the only span incident to v_k is $S(bv_{k-1}v_1)$, with c as replacement vertex. But $\rho v_{k-1} = 4$ and $\rho c \geq 5$. Therefore v_k is a good vertex.

II.2.2 $k = 2$

Then the only asymmetric span incident to v_2 is $S(bav_1)$, with c as replacement vertex. However, $\rho a = 5$ and $\rho c \geq 5$. Therefore v_2 is a good vertex.

<u>Case III S is of type III</u>

Then, the only asymmetric span incident to v_3 is $S(v_2v_1b)$, with a as the replacement vertex. However in G, $\rho v_1 = 5$, and $\rho a \geq 5$. Therefore v_3 is a good vertex.

<u>Case IV S is of type IV</u>

Then the only span incident to v_3 in G is $S(v_2ab)$, with v_1 as replacement vertex. But in G, $\rho a > 5$ and $\rho v_1 = 4$. Therefore v_3 is a good vertex.

<u>Case V S is of type V</u>

We consider three cases.

<u>V.1</u> <u>The pivot vertices of S are a and b</u>

Then in G, $\rho c = 4$ and $\rho a \geq 6$. But the only span incident to v_2 is $S(v_1ab)$, with c as replacement vertex. Hence v_2 is a good vertex.

<u>V.2</u> <u>The pivot vertices of S are c and b</u>

Therefore $\rho a = 5$ and $\rho c \geq 5$ in G. Thus we again have that v_2 is a good vertex.

<u>V.3</u> <u>The pivot vertices of S are c and a</u>

Then $\rho b = 6$ and $\rho a > 5$. But the only span incident to v_3 is $S(v_1bv_2)$, with a as replacement vertex. Therefore v_3 is a good vertex.

<u>Case VI S is of type VI</u>

Then in G, $\rho b > 5$ and $\rho v_1 = 4$. But the only asymmetric span incident to v_3 is $S(v_2ba)$, with replacement vertex v_1. Therefore v_3 is a good vertex.

<u>Case VII S is of type VII</u>

Then $\rho v_1 = 5$ and $\rho a > 6$ in G, and the only span incident to v_4 is $S(v_3av_2)$ with v_1 as replacement vertex. Therefore v_4 is a good vertex.

Case VIII S is of type VIII

Then the only span incident to v_4 in G is $S(v_3v_1v_2)$, with b as replacement vertex. But $\rho v_1 = 6$ and $\rho b > 5$ in G, therefore v_4 is a good vertex.

This final case concludes the proof of the theorem. We now give the table which summarizes the above proof.

Type of S	Pivot vertices of S	Good Vertex	Asymmetric span S' incident to the good vertex	Valency in G of replaced vertex of S'	Replacement vertex of S' and its valency in G
Type I	a and c (or b and c)	v_k	$S'(bv_{k-1}a)$	$\rho v_{k-1} = 4$	u; $\rho u \geq 4$
Type II (k=2)	a and b	v_2	$S'(v_1ab)$	$\rho a > 5$	c; $\rho c = 4$
	b and c	v_2	$S'(v_1ab)$	$\rho a = 5$	c; $\rho c \geq 5$
Type II (k≥3)	a and b	v_k	$S'(v_1v_{k-1}b)$	$\rho v_{k-1} = 4$	c; $\rho c = 4$
	b and c	v_k	$S'(v_1v_{k-1}b)$	$\rho v_{k-1} = 4$	c; $\rho c \geq 5$
Type III	any pair	v_3	$S'(v_2v_1b)$	$\rho v_1 = 5$	a; $\rho a \geq 5$
Type IV	any pair	v_3	$S'(v_2ab)$	$\rho a > 5$	v_1; $\rho v_1 = 4$
Type V	a and b	v_2	$S'(v_1ab)$	$\rho a \geq 6$	c; $\rho c = 4$
	c and b	v_2	$S'(v_1ab)$	$\rho a = 5$	c; $\rho c \geq 5$
	c and a	v_3	$S'(v_1bv_2)$	$\rho b = 6$	a; $\rho a > 5$
Type VI	any pair	v_3	$S'(v_2ba)$	$\rho b > 5$	v_1; $\rho v_1 = 4$
Type VII	any pair	v_4	$S'(v_3av_2)$	$\rho a > 6$	v_1; $\rho v_1 = 5$
Type VIII	any pair	v_4	$S'(v_3v_1v_2)$	$\rho v_1 = 6$	b; $\rho b > 5$

□

SECTION 5.3 - RECONSTRUCTION

Proof of the Main Theorem of Chapter 5

Since, as we have seen, every maximal planar graph whose minimum valency is at least 4 is vertex-reconstructible, then we need only consider maximal planar graphs with minimum valency 3. Moreover, as noted in the beginning of this chapter, since the maximal planarity of such graphs is recognizable from the vertex-deck, then we need only consider the vertex-reconstruction of collapsible graphs. Thus, let G be a collapsible graph. By Theorem 5.5 we know that G has a good vertex v_0. But then, by Theorem 5.2 and the definition of a good vertex, $G - v_0$ has exactly two non-equivalent ρv_0-representations R and R', with the property that the family of valencies of all the vertices incident to the ρv_0-face of R is different from the family of valencies of the vertices incident to the ρv_0-face of R'. But then, since we know the neighbourhood valency list of v_0 in G, we can reconstruct uniquely from $G - v_0$. □

PART III EDGE-RECONSTRUCTION

In this part we show that certain classes of planar graphs are edge-reconstructible. We start, in Chapter 6, by showing that planar graphs with minimum valency 5 are edge-reconstructible. Then in Chapter 7 we show that 4-connected planar graphs are edge-reconstructible. In this chapter heavy use is made of the concept of reconstructor sets and reconstructor sequences, which we introduced in Chapter 3. We also make extensive use of the technique involving associates which we employed in Theorem 3.3 to show that R_2 reconstructs $\mathcal{J}$. In the last section of Chapter 7 we present a brief discussion on the reconstruction of graphs from edge-contracted subgraphs, a problem which in certain cases can be regarded as dual to the Edge-reconstruction Problem. In this section we show that 3-connected bipartite graphs and maximal planar graphs are reconstructible from their edge-contracted subgraphs.

CHAPTER 6 PLANAR GRAPHS WITH MINIMUM VALENCY 5

In [F1], Fiorini showed that 4-connected planar graphs with minimum valency 5 are edge-reconstructible. The aim of this chapter is to remove completely the restriction on connectivity.

MAIN THEOREM OF CHAPTER 6

Planar graphs with minimum valency 5 are edge-reconstructible.

We first note that the question of recognition is resolved by the following, a proof of which can be found in [F1].

<u>Theorem 6.1</u>

A connected graph of order at least 7 and minimum valency at least 3 is planar if and only if every edge-deleted subgraph is planar. □

Furthermore, we shall make use of the following theorem, also proved in [F1].

<u>Theorem 6.2</u>

If G is a 3-connected plane graph, with minimum valency 5, then either G contains two adjacent 5-vertices or else G contains a 5-vertex incident only to 3-faces. □

Separable and disconnected graphs with minimum valency 5 are edge-reconstructible. This holds since all disconnected graphs and all separable graphs with no 1-vertices are vertex-reconstructible (see [BH1]), and by Theorem 3.2, if a graph with no isolates is vertex-reconstructible then it is also edge-reconstructible. We are thus left with the consideration of planar graphs with minimum valency 5 and which are either 3-connected or have connectivity 2. We consider these two cases in Sections 6.1 and 6.2 respectively.

SECTION 6.1 - CONNECTIVITY AT LEAST 3

Theorem 6.3

Let G be a 3-connected plane graph with minimum valency at least 4, and let v be a 5-vertex of G incident only to 3-faces. Then there exists an edge e incident to v such that G_e is 3-connected.

Proof

Let the boundary circuits of the 3-faces incident to v be $vv_iv_{i+1}v$, $i = 0,1,2,3,4$ (modulo 5). Let e_i be the edge vv_i and let us consider $G - e_0$. If $G - e_0$ is 3-connected, then we have nothing to prove. Therefore we assume that $G - e_0$ has connectivity 2, so that there exists a separating set of vertices $\{x_1,x_2\}$ in $G - e_0$, which is not a separating set in G. Clearly, $\{v,v_0\}\cap\{x_1,x_2\} = \emptyset$, and also $\{x_1,x_2\}$ separates v and v_0 in $G - e_0$, since $\{x_1,x_2\}$ is not a separating set in G. Therefore $\{x_1,x_2\} = \{v_1,v_4\}$.

Now, let H be that component of $(G - e_0) - \{v_1,v_4\}$ which contains the vertex v_0. Clearly, v_0 cannot be the only vertex of H, because otherwise its valency in G would be 3. Therefore let $w \in VH$, $w \neq v_0$; w and v are separated in G by $\{v_4,v_0,v_1\}$.

Since G is 3-connected, then by Theorem 2.2, there exist in G, three internally disjoint chains $C_1 = C_1[v,w]$, $C_2 = C_2[v,w]$, and $C_3 = C_3[v,w]$, and since $\{v_4,v_0,v_1\}$ separates v and w in G, we may assume that $v_4 \in VC_1$, $v_0 \in VC_2$ and $v_1 \in VC_3$. We may also assume e_4, e_0 and e_1 are edges of C_1, C_2 and C_3 respectively.

We shall now show that $G - e_1$ is 3-connected. We assume the contrary and derive a contradiction. Since $G - e_1$ is not 3-connected, then as above, $\{v_0,v_1,v_2\}$ is a separating set for G. Therefore there exists a vertex w', such that v and w' are separated by $\{v_0,v_1,v_2\}$ in G. We let C_1', C_2', C_3' be three internally disjoint chains from v to w' in G, such that e_0, e_1, e_2 are edges of C_1',

C_2', C_3' respectively.

We first note that $w \neq w'$, because otherwise we obtain a contradiction as follows. Since C_3' and C_1' are internally disjoint, then C_3' does not contain e_0. Therefore C_3' exists in $G - e_0$. But C_3' and C_2' are internally disjoint, and hence C_3' does not contain v_1, so that C_3' contains v_4, since v and w are separated in $G - e_0$ by $\{v_4,v_1\}$. Thus if $C_3' = vv_2s_1s_2\ldots s_rw$, then v_4 is s_j for some j. Therefore $vs_js_{j+1}\ldots s_rw$ is a chain from v to w, passing through neither one of v_0, v_1, v_2, contradicting the fact that v and w' (=w) are separated by $\{v_0,v_1,v_2\}$ in G. We conclude that $w' \neq w$.

Now, let $\Gamma_1 = \{C_1 \cup C_3\} - v$, and let $\Gamma_2 = \{C_1' \cup C_3'\} - v$. Since G is planar, $V\Gamma_1 \cap V\Gamma_2 \neq \emptyset$. Let p be the first vertex on Γ_1 (as traversed from v_4 to v_1) that lies also on Γ_2 (we are not excluding the possibility that $p = v_4$). Let Γ_1' be that part of Γ_1 between v_4 and p (inclusive) and let Γ_2' be that part of Γ_2 between p and w' (inclusive). Therefore $\Gamma_1' \cup \Gamma_2' \cup \{e_4\}$ is a chain from v to w', passing through none of the vertices v_0, v_1, v_2, contradicting the fact that v and w' are separated by $\{v_0,v_1,v_2\}$ in G. □

Theorem 6.4

All 3-connected planar graphs of minimum valency 5 are edge-reconstructible.

Proof

Let G be a 3-connected planar graph with minimum valency 5. The planarity of G is recognizable from its edge-deck by Theorem 6.1 (we may assume that the order of G is at least 7, since in fact a planar graph with minimum valency 5 must have at least twelve vertices). Moreover, we may assume that no two 5-vertices of G are adjacent, since if u and v are two adjacent 5-vertices, then G is uniquely reconstructible from G_{uv}. Therefore by Theorem 6.2, in the plane embedding of G, there is a vertex w of valency 5, such that w is incident only to 3-faces. Thus, by Theorem 6.3, there exists an edge e_0 incident to w, such that $G - e_0$ is 3-connected, and hence has a unique plane representation. In this plane representation, the vertex w has valency 4 and is incident to three 3-faces and one 4-face F. We can therefore reconstruct G uniquely from $G - e_0$ by joining the unique 4-vertex w of $G - e_0$ to the unique vertex, incident to F, to which w is not already adjacent. □

SECTION 6.2 - CONNECTIVITY 2

In this section we shall show that planar graphs with connectivity 2 and minimum valency 5 are edge-reconstructible. The proof we give here is an improved version of our earlier work [J. Graph Theory, Vol.3, pp. 273-285].

Euler's formula for the plane states that if K is a plane graph with ν vertices, ε edges and ϕ faces, then

$$\nu + \phi = \varepsilon + 2.$$

This formula also holds if we allow the possible existence of multiple edges. Moreover, in this case, if the boundary of each face of K is a circuit with at least three edges, we have that

$$3\phi \leq 2\varepsilon,$$

which, together with Euler's formula, yields the usual inequality,

$$\varepsilon \leq 3\nu - 6.$$

Therefore this inequality also holds for plane general graphs, provided that the above condition on the sizes of circuits bounding faces holds. (We would like to remind the reader that any graph considered will be simple unless otherwise specified.)

Lemma 6.1

Let K be a plane general graph in which the boundary of every face is a circuit with at least three edges. Also let $\rho a \geq 2$ and $\rho b \geq 2$ for two vertices $a, b \in VK$, and let $\rho v \geq 5$ for any vertex $v \in VK - \{a,b\}$. Then there exist at least four 5-vertices in $VK - \{a,b\}$.

Proof

Let k be the number of 5-vertices in $VK - \{a,b\}$, and let $\varepsilon = \varepsilon K$ $\nu = \nu K$. Therefore $6(\nu - k - 2) + 5k + 2 \cdot 2 \leq 2\varepsilon$. But by the remarks above, $\varepsilon \leq 3\nu - 6$, so that $k \geq 4$. □

The next lemma is obvious and its proof is omitted.

Lemma 6.2

Let G be a graph with connectivity κ, let Q be a separating κ-set of G, and let the components of $G - Q$ be $H_1, H_2, \ldots, H_r$. Then $C(G,\overline{H}_i) = Q$. □

We now have a few definitions. Let G be a planar graph, let $\kappa G = 2$, and let $Q = \{a,b\}$ be a separating set of G such that the components of $G - Q$ are $H_1, H_2, \ldots, H_r$. Then each $\overline{H}_i$, $i = 1,2,\ldots,r$ will be called a lobule of G. Let

Let L = {H: H is a lobule of G, C(G,H) = {a,b}, for all separating pairs {a,b} of G}.

By a minimal lobule of G we mean a lobule of minimal order, where minimality is taken over all L.

Lemma 6.3

Let H be a minimal lobule of a planar graph G of minimum valency 3 and connectivity 2, and let $C(G,H) = \{a,b\}$. Then

(i) H is 2-connected;

(ii) in H, $\rho a, \rho b \geq 2$;

(iii) if $Q = \{u,v\}$ is a separating pair for H, then Q cannot be a separating pair for G.

Proof

(i) $H - \{a,b\}$ is connected by definition, so that H must also be connected; otherwise at least one of a or b (say a), is not adjacent to any vertex of $H - \{a,b\}$. But then $\{b\}$ would be a separating set for G, which is impossible.

We must now show that H is not separable. Let us assume the contrary, and let x be a separating vertex for H. Since G is 2-connected, $H - x$ can have only two components, L_1 and L_2, with a and b in different components, so that $x \notin \{a,b\}$. We therefore have that in H, $\rho x \geq 3$, and we may assume that $a \in VL_1$ and $b \in VL_2$. Since in H, $\rho x \geq 3$, we may assume that $\nu\overline{L}_1 > 2$. Therefore there exists at least one vertex z in $VL_1 - a$. Since z is in L_1 and b is in L_2, then any chain in H from z to b must pass through x. Also, in G, any chain from z to a vertex not in H must pass through either of a or b. Therefore in G, any chain from z to a vertex not in $\overline{L}_1$ must contain either a or x. Therefore $\{a,x\}$ is a separating pair for G, and $\overline{L}_1$ is a lobule of G, contradicting the minimality of H.

(ii) This follows from (i).

(iii) Let the components of $H - \{u,v\}$ be $L_1, L_2, \ldots, L_r$. If $\{u,v\}$ were a separating pair for G, then at least one of $\overline{L}_i$ would be a lobule of G, contradicting the minimality of H. □

Lemma 6.4

Let G be a planar graph with minimum valency 5 and connectivity 2, in which no two 5-vertices are adjacent. If H is a minimal lobule of G with $C(G,H) = \{a,b\}$, then in any plane representation of H there are at least four 5-vertices in $VH - \{a,b\}$ which are incident solely to 3-faces.

Proof

We note first that by Lemma 6.3(ii), the valencies of a and b in H are at least 2, so that by Lemma 6.1, there are at least four 5-vertices in $VH - \{a,b\}$. Let R be a plane representation of H, and let v be a 5-vertex of H, $v \in VH - \{a,b\}$, such that v is not incident solely to 3-faces in R. Then we can find a k-face F in R, with boundary circuit $xvys_1s_2\ldots s_{k-3}x$, $k > 3$. Therefore by adding an edge vs_i inside the face F, the vertex v becomes a 6-vertex, and the resulting plane graph is either simple or, if not, it still has the property that the boundary of every face is a circuit with at least three edges. We note that since no two 5-vertices of G are adjacent, then neither x nor y can be five vertices of H different from a and b. Hence, this process can be repeated for each 5-vertex of H in $VH - \{a,b\}$ which is incident in R to some t-face, $t > 3$. At each step, at least one 5-vertex from $VH - \{a,b\}$ has its valency increased to 6, and we obtain a new plane graph which might have multiple edges but in which the boundary of each face is still a circuit with at least three edges. Therefore by Lemma 6.1, this process must fail for at least four of the 5-vertices in $VH - \{a,b\}$, that is, at least four of these 5-vertices are incident solely to 3-faces in R. □

We shall require the following lemma which is proved as Theorem 2.1.1 in [O1].

Lemma 6.5

Let B be a bridge of a circuit C in G, and let a_1, a_2, a_3 be three vertices of attachment of B with C. Then there exists a vertex v_0 in B, v_0 not a vertex of attachment, such that in B there exist three internally disjoint chains $C[v_0,a_i]$, $i = 1, 2, 3$. □

Lemma 6.6

Let G be a 2-connected planar graph and let R be a plane representation of G. Let v be a 5-vertex of G such that, in R, v is incident solely to the 3-faces with boundary $vv_iv_{i+1}v$, $i = 0,1,2,3,4$ (modulo 5). Then there exists another plane representation of G in which v is incident to a non-triangular face if and only if at least one pair $\{v_i,v_{i+1}\}$ is a separating set for G.

Proof

Let us assume first that there exists a plane representation R' of G in which v is no longer incident solely to 3-faces. Then at least one of the circuits $vv_iv_{i+1}v$, for some i, is no longer a boundary circuit of a face in R'. We may assume that $T = vv_1v_2v$ is not the boundary of a face in R'. Therefore T must have at least two bridges B_1 and B_2. We first show that not both B_1 and B_2 can have three vertices of attachment with T; otherwise, by Lemma 6.5, we can find $w_1 \in VB_1 - VT$, $w_2 \in VB_2 - VT$, and six internally disjoint chains $C[w_j,v]$ and $C[w_j,v_i]$, $i,j \in \{1,2\}$. Therefore T can never be the boundary of a face of G, a contradiction. Therefore we may assume that B_1 has only two vertices of attachment with T (B_1 must have at least two vertices of attachment since G is 2-connected). We now show that B_1 cannot have v as a vertex of attachment with T. Otherwise, at least one of the edges vv_i, $i=3,4,0$, is in B_1. But each of these edges is joined by a T-avoiding chain to each of the edges v_0v_1 and v_3v_2, implying that both v_1 and v_2 are vertices of attachment of B_1 with T, a contradiction. Therefore the vertices of attachment of B_1 with T are v_1 and v_2. Hence $\{v_1,v_2\}$ is a separating set for G.

Conversely, let us assume that $\{v_1,v_2\}$ is a separating set for G. Then there exists a vertex $w \notin VT$, such that any chain from w to v must pass through at least one of v_1 or v_2. Moreover, since G is 2-connected, we can find two internally disjoint chains from w to v, hence we can find two internally disjoint chains $C_1 = C_1[w,v_1]$ and $C_2 = C_2[w,v_2]$ not containing v. Now, both these chains are in the same bridge B of T. Moreover, this bridge cannot have v as a vertex of attachment with T, as otherwise we could find a chain from w to v not passing through either of v_1 or v_2. We deduce that the vertices of attachment of B with T are v_1 and v_2. Therefore if in R we transfer B from ExtT to IntT (or vice-versa), we obtain a representation of G in which T is not the boundary of a face. □

Lemma 6.7

Let G be a planar graph with connectivity 2, and let H be a lobule of G with $C(G,H) = \{a,b\}$. If the vertices $x,y \in VH$ are such that the set $\{x,y\}$ is a separating pair for G but not for H, then $\{x,y\} = \{a,b\}$.

Proof

Let K be the graph induced by the vertex-set $\{VG - VH\} \cup \{a,b\}$. Then K is connected. We assume that $\{x,y\} \neq \{a,b\}$ and derive a contradiction. We consider two cases.

Case 1 $\{x,y\} \cap \{a,b\} = \emptyset$

Since $\{x,y\}$ is not a separating pair for H, then for any vertex $v \in VH - \{x,y\}$, there exists a chain in H from v to a, passing through neither x nor y. Also, since x, y are not in VK, then for any w in VK, there exists a chain in K from w to a, passing through neither x nor y. Therefore $G - \{x,y\}$ is connected (since $a \in VK \cap VH$), a contradiction.

Case 2 $\{x,y\} \cap \{a,b\} \neq \emptyset$

We can therefore assume that $y = b$ and $x \neq a$. Again, since $\{x,y\}$ ($= \{x,b\}$) is not a separating pair for H, then for any $v \in VH - \{x,b\}$ there exists a chain from v to a, passing through neither x nor b. Also, since $x \notin VK$, then for any vertex w in VK, there exists a chain in K, from w to a, not passing through x. Hence, if for any w in $VK - \{b\}$ there exists a chain in K from w to a not passing through b, we would deduce, as in Case 1, that $G - \{x,b\}$ is connected. Therefore there exists a vertex w_0 in $VK - \{b\}$, such that any chain in K from w_0 to a must pass through b. It follows that $\{b\}$ is a separating set for K, and hence for G, since $C(G,H) = \{a,b\}$ (that is, since any new $[w_0,a]$-chain in G which does not exist in K must also contain b). But this contradicts the fact that G is 2-connected. This final contradiction establishes the result. □

Theorem 6.5

Let G be a planar graph with minimum valency 5, connectivity 2, and such that no two 5-vertices of G are adjacent. Then there exists a 5-vertex v of G such that, in any plane representation of G, v is incident only to 3-faces.

Proof

Let H be a minimal lobule of G, and let $C(G,H) = \{a,b\}$. To any plane representation of G there corresponds a plane representation of H, in which a and b are incident to a common face. Let R be any such plane representation of H. Then by Lemma 6.4 there exist at least four 5-vertices in $VH - \{a,b\}$ which are incident solely to 3-faces in R. Let x, y, z be three such vertices, and let x

be incident in R to the faces with boundary circuits $xx_ix_{i+1}x$, $i = 0,1,2,3,4$ (modulo 5). Hence $Nx = \{x_0,x_1,x_2,x_3,x_4\}$. In a similar fashion let $Ny = \{y_i : 0 \le i \le 4\}$ and $Nz = \{z_i : 0 \le i \le 4\}$. We now claim that at least one of x, y or z is incident solely to 3-faces in any plane representation of G. We first note that each one of x, y, z is, in at least one plane representation of G, incident solely to 3-faces, since a and b, the vertices of contact of H in G are incident to a common face in R. Thus, if we assume that there exists some plane representation of G in which x is not incident solely to 3-faces, it follows from Lemma 6.6 that $\{x_i,x_{i+1}\}$ is a separating pair for G, for some i, $0 \le i \le 4$. But then, by Lemma 6.3(iii), $\{x_i,x_{i+1}\}$ cannot be a separating pair for H, so that by Lemma 6.7, $\{x_i,x_{i+1}\} = \{a,b\}$. Similarly, if there exists a plane representation of G, in which y is not incident solely to 3-faces, then $\{y_j,y_{j+1}\} = \{a,b\}$ for some j. But then $\{a,b\}$ cannot be $\{z_k,z_{k+1}\}$ for any k, because otherwise the edge ab would be incident in R to the three faces bounded by the circuits abxa, abya, abza, since for all t, $0 \le t \le 4$, the circuits $xx_tx_{t+1}x$, $yy_ty_{t+1}y$, $zz_tz_{t+1}z$ bound faces in R. Since this is impossible, it follows that z is incident solely to 3-faces in any plane representation of G. □

<u>Theorem 6.6</u>

Let G be a planar graph with minimum valency 5 and connectivity 2. Then G is edge-reconstructible.

<u>Proof</u>

The planarity of G is recognizable from its edge-deck, by Theorem 6.1. Moreover, we can assume that no two 5-vertices of G are adjacent, as otherwise edge-reconstruction is trivial. Therefore by Theorem 6.5 there exists a 5-vertex v of G, such that in any plane representation of G, v is incident only to 3-faces. We now claim that there exists an edge e of G, such that G is uniquely edge-reconstructible from G_e. In fact this edge will be one of the edges incident to v.

Let the 3-faces incident to v have boundary circuits $vv_iv_{i+1}v$, $i = 0,1,2,3,4$ (modulo 5), and let e_i be the edge vv_i. We note that $G - e_0$ has at least one plane representation in which v, the only 4-vertex of $G - e_0$, is incident to the three 3-faces with boundary circuits $vv_iv_{i+1}v$, $i = 1,2,3$, and the 4-face F, bounded by the circuit $C = vv_4v_0v_1v$. In any plane representation of $G - e_0$, the 3-circuits $vv_iv_{i+1}v$, $i = 1,2,3$, are always the boundary circuits of faces, since they are always the boundaries of faces in any plane representation of G. Therefore ambiguity in reconstructing G from $G - e_0$ can only arise if there exists a plane representation of $G - e_0$ in which C does not bound a face. We may therefore assume that C has at least two bridges, B_1 and B_2, in $G - e_0$. We first observe that B_1 and B_2 cannot both have v as vertex of attachment with C. If we assume the contrary, then as in the proof of Lemma 6.6, both v_4 and v_1 would be vertices of attachment of B_1 and B_2 with C; but then, applying Lemma 6.5 to B_1 and B_2 for the vertices of attachment v_4, v_0, v_1, we conclude that C can never be the boundary of a face in $G - e_0$, a contradiction. We may therefore assume that B_1 does not have v as vertex of attachment with C. Moreover, if v_4 is not a vertex of attachment of B_1 with C, then there exists a plane representation of G in which vv_1v_0v is not a boundary of a face, which is impossible, since v is incident solely to 3-faces in any plane representation of G. Hence v_4 is a vertex of attachment of B_1 with C, and similarly, v_1 is a vertex of attachment of B_1 with C. We can therefore find a chain $C_1 = C_1[v_4,v_1]$, whose internal vertices are all in $VB_1 - VC$.

We shall now show that G is uniquely reconstructible from $G - e_1$. Thus, let us consider $G - e_1$. Again, as for $G - e_0$, we obtain that $G - e_1$ has a plane representation in which v, the only 4-vertex, is incident to three 3-faces and one 4-face F', bounded by the circuit $C' = vv_0v_1v_2v$. As above, ambiguity in reconstructing G from $G - e_1$ can only arise if there exists a plane representation of $G - e_1$ in which C' does not bound a face. Therefore if we assume that G is not uniquely reconstructible from $G - e_1$, then as before, there exists a bridge B_1' of C', such that B_1' has v_0 and v_2, but not v, as vertices of attachment with C'. It follows that there exists a chain $C_1' = C_1'[v_0,v_2]$ whose internal vertices are all in $VB_1' - VC'$. But G is planar, so that C_1 and C_1' cannot be disjoint. Therefore v_2 is in B_1, and hence B_1 has v as vertex of attachment with C in $G - e_0$, a contradiction. □

This theorem, together with Theorem 6.4, completes the proof of the Main Theorem of this chapter.

CHAPTER 7 PLANAR GRAPHS WITH CONNECTIVITY AT LEAST 4

In the previous chapter we extended Fiorini's work in [F1] on the edge-reconstruction of 4-connected planar graphs with minimum valency 5 by removing the condition on connectivity. In this chapter we shall extend Fiorini's result in another direction by relaxing the condition on the minimum valency.

MAIN THEOREM OF CHAPTER 7

Every 4-connected planar graph is edge-reconstructible.

In view of [F1] (or Theorem 6.4) there remains to show that 4-connected planar graphs with minimum valency 4 are edge-reconstructible. In Sections 7.1 and 7.2, $\mathfrak{J}_0$ will denote the class of all such graphs, and G will be a graph in $\mathfrak{J}_0$. (The question of edge-recognition is again settled by Theorem 6.1.) As usual we shall also assume that no two 4-vertices are adjacent in G, as otherwise G would be trivially edge-reconstructible.

Since every graph in $\mathcal{D}'G$ is 3-connected, we shall not encounter any problems about non-equivalent plane representations, unlike the situations we were faced with in Chapter 5 and Chapter 6 (§. 6.2). In fact, as we said in Chapter 2, we may assume that G and all the graphs in $\mathcal{D}'G$ are plane graphs, and in particular, we can talk about the face-valency list of each of these graphs.

We shall need the following theorem on plane graphs, which is analogous to, but slightly more involved than Theorem 6.2.

<u>Theorem 7.1</u>

Either G contains a 5-vertex incident to no k-face, $k \geq 6$, or else G contains a 4-vertex incident to at least three triangular faces.

Proof

We assume that the theorem is false and derive a contradiction. By our assumption, every 5-vertex of G is incident to at least one k-face, $k \geq 6$, and every 4-vertex is incident to at least two non-triangular faces. For every 5-vertex of G, we choose a k-face, $k \geq 6$, incident to the 5-vertex, and insert a vertex inside this face, joining it to all the vertices incident to this face, giving a plane graph G'. Now, in G', no two 4-vertices and no two 5-vertices are adjacent (since these were 4-vertices in G), and similarly no 5-vertex is adjacent to a 4-vertex. Moreover, any 4-vertex of G' is still incident to at least two non-triangular faces, and every 5-vertex is incident to at least one non-triangular face. We now construct G'' from G' in the following way. We call any 4-vertex or 5-vertex of G' a small vertex. If F is a k-face, $k \geq 4$, to which at least three small vertices are incident, we introduce a circuit in F passing through the successive small vertices. If F has exactly two small vertices incident to it we join them by an edge, whereas if F has only one small vertex incident to it, we join this small vertex to another vertex, incident to F, and to which the small vertex is not already adjacent. In view of the above forbidden adjacencies, and in view of the fact that every 5-vertex of G' is incident to at least one non-triangular face and every 4-vertex is incident to at least two non-triangular faces, then the graph G'' so constructed is a plane graph with minimum valency greater then 5, a contradiction. □

Our main efforts in this chapter will be devoted to proving the following theorem which, together with Theorem 7.1, will clearly imply the Main Theorem.

Theorem 7.2

If G is not edge-reconstructible then every 5-vertex of G is incident to at least one k-face, $k \geq 6$, and every 4-vertex is incident to at least two non-triangular faces.

Before proceeding to the proof of Theorem 7.2 we have to introduce some new notation and results in Section 7.1. Section 7.2 will deal more directly with the proof of Theorem 7.2.

SECTION 7.1 - FACE-VALENCIES, WHEEL-SEQUENCES AND RECONSTRUCTOR SEQUENCES

We note first that if an edge e is incident to faces F and F' in G, then the new face resulting from the deletion of e has valency $\rho^*F + \rho^*F' - 2$ in $G - e$. Since both ρ^*F and ρ^*F' are at least 3, then each is strictly less than $\rho^*F + \rho^*F' - 2$.

Given a graph $G - e$ in $\mathcal{D}'G$, we say that the face F of $G - e$ is a root-face if we can determine that F is not a face in G, that is, that the missing edge e should be inside the face F. For example, if $G - e_0$ in $\mathcal{D}'G$ has the property that $\Delta^*(G-e_0) \geq \Delta^*(G-e)$ for all $G - e$ in $\mathcal{D}'G$, then the face in $G - e_0$ with face-valency equal to $\Delta^*(G - e_0)$ is a root-face.

Now, since maximal planar graphs are edge-reconstructible, we can assume that $\Delta^*G \geq 4$. We then order the graphs in $\mathcal{D}'G$ as $G - e_1, G - e_2, \ldots, G - e_\varepsilon$ such that $\Delta^*(G-e_i) \geq \Delta^*(G-e_{i+1})$, and it is then easily seen that for $i = 1,2,3,4$, the face F_i with face-valency equal to $\Delta^*(G-e_i)$ is a root-face.

Theorem 7.3

The face-valency list of G is reconstructible from $\mathcal{D}'G$.

Proof

Let F_i and $G - e_i$, $i = 1,2,3,4$, be as above. With each of these four $G - e_i$ we associate the list L_i of face-valencies of all the faces of $G - e_i$ except F_i. Let A_i and B_i be the faces incident to e_i in G; therefore $\rho^*A_i + \rho^*B_i - 2 = \rho^*F_i$. Hence L_i contains all the face-valencies of G except for the face-valencies of the two faces A_i and B_i. If for some i we can determine ρ^*A_i or ρ^*B_i, we then have the required result.

Let us assume first that for some $i \neq j$, $\{i,j\} \subset \{1,2,3,4\}$, $L_i \neq L_j$. Then there exists a positive integer x which appears p times in L_i and at most $p - 1$ times in L_j, $p \geq 1$. But then x is one of ρ^*A_j or ρ^*B_j, from which the required result follows.

We may therefore assume that $L_i = L_j$ for all $i \neq j$, $\{i,j\} \subset \{1,2,3,4\}$. Hence $\{\{\rho^*A_i,\rho^*B_i\}\} = \{\{\rho^*A_j,\rho^*B_j\}\}$ for all $i \neq j$, so that $\rho^*F_i = \rho^*F_j = \Delta^*$, say. We now augment the family $\{\{G-e_1,\ldots,G-e_4\}\}$ to $\{\{G-e_i: i = 1,2,3,\ldots,q\}\}$, $q \geq 4$, by adding any other graphs from $\mathcal{D}'G$ which have maximum face-valency equal to Δ^*. As for $G-e_1,\ldots,G-e_4$, the Δ^*-face F_i of $G - e_i$, $5 \leq i \leq q$ (if any such graphs do exist), is a root-face. With each of these graphs $G - e_i$ we again associate the list L_i and the pair of faces A_i, B_i. As before we may assume that $L_i = L_j$, whenever $i \neq j$, $\{i,j\} \subset \{1,\ldots,q\}$, so that $\{\{\rho^*A_i,\rho^*B_i\}\} = \{\{\rho^*A_j,\rho^*B_j\}\}$.

We then search for a graph $G - e_i$, $1 \leq i \leq q$, in which there are two adjacent faces A and B, such that neither A nor B is F_i, and such that $\rho^*A + \rho^*B - 2 = \rho^*F_i$. But then $\{\{\rho^*A,\rho^*B\}\} = \{\{\rho^*A_i,\rho^*B_i\}\}$, and the problem is solved. We may therefore assume that no such $G - e_i$ exists, so that any two pairs $\{A_i,B_i\}$ and $\{A_j,B_j\}$ have at least one face in common. Since the connectivity of G is greater than 2, any two such pairs can only have one common face, and hence, since there are at least four such pairs, it follows that there is a face common to each pair. We may therefore assume that $B_i = B_j = B$, say, and $A_i \neq A_j$, for all $i \neq j$. That is, B is adjacent to all the q faces $A_1, A_2, \ldots, A_q$.

We now look for a graph $G - e_k$ in $\mathcal{D}'G - \{\{G-e_i: i = 1,2,\ldots,q\}\}$ such that in $G - e_k$ there is a face B' adjacent to q faces $A_1', A_2', \ldots, A_q'$, and such that $\rho^*B' + \rho^*A_i' - 2 > \Delta^*$, for each $1 \leq i \leq q$. Then B' cannot be a face in G, and so B' is a root-face. We then write the list L_k of face-valencies of all faces in $G - e_k$ except that of B'. Since $\rho^*B' < \Delta^*$, we obtain that $L_k \neq L_i$, for any $i = 1,2,\ldots,q$, and so we can continue as above (that is, by comparing L_k with one of the L_i, $i = 1,2,\ldots,q$).

We may therefore assume that no such $G - e_k$ exists. But then we have that in G, the face B is adjacent only to the faces A_i, $i = 1,2,\ldots,q$, and therefore $\rho^*B = q$. □

<u>Corollary 7.1</u>

Let $G - e$ be any graph in $\mathcal{D}'G$ and let F and F' be the two faces of G incident to e. Then the pair $\{\{\rho^*F, \rho^*F'\}\}$ is reconstructible from $\mathcal{D}'G$.

<u>Proof</u>

Let L be the list of all face-valencies of G (which list we have just reconstructed in Theorem 7.3) and let L' be the list of all face valencies in $G - e$. Then $\rho^*F = \rho^*F' = x$, say, if and only if x appears in L twice more than it appears in L'; $\rho^*F = x \neq y = \rho^*F'$ if and only if each one of x and y appears in L once more than it does in L'. □

Corollary 7.1 will turn out to be crucial in all that follows.

Let K be a 3-connected plane graph, and let v be a k-vertex of K. Let the faces incident to v be $F_0, F_1, \ldots, F_{k-1}$ such that F_i is adjacent to F_{i-1} and F_{i+1} (modulo k), and let F_i be an (a_i+2)-face. Then $W(v) = \langle a_0, a_1, \ldots, a_{k-1} \rangle$ is called the <u>wheel-sequence</u> of v in K. Each a_i is called a <u>term</u> of the wheel-sequence. We note that the wheel-sequence of v is unique up to choice of initial term and orientation. A wheel-sequence with k terms is sometimes called a <u>k-sequence</u>. The <u>rim-length</u> of $W(v)$ is equal to $\sum_{i=0}^{k} a_i$.

Let $\mathcal{W}(v)$ be that subgraph of K induced by all the edges which are incident to the faces $F_0, F_1, \ldots, F_{k-1}$. Then $\mathcal{W}(v)$ is called the <u>wheel</u> of v in K. Let the neighbours of v be $v_0, v_1, \ldots, v_{k-1}$ such that the edge incident to both F_i and F_{i+1} (modulo k) is vv_i. Then the chain consisting of all those edges (not incident to v) and all those vertices (apart from v) which are incident to the face F_i is called the <u>$\mathcal{W}(v)$-chain from v_{i-1} to v_i</u>.

To prove Theorem 7.2 we have to show that if G is not edge-reconstructible, then every 5-vertex has in its wheel-sequence at least one term greater than 3, and every 4-vertex has in its wheel-sequence at least two terms greater than 1. With this terminology we can reformulate Theorem 7.2 as follows.

<u>Theorem 7.2'</u>

If G contains a 5-vertex with wheel-sequence $\langle a_0, a_1, a_2, a_3, a_4 \rangle$ such that $a_i \le 3$ for all $0 \le i \le 4$, or a 4-vertex with wheel-sequence $\langle 1,1,1,a \rangle$, $a \ge 1$, then G is edge-reconstructible.

We shall prove Theorem 7.2' by exhibiting a reconstructor set for $\mathcal{J}_0$ which includes all 5-sequences $\langle a_0, a_1, a_2, a_3, a_4 \rangle$, $a_i \le 3$. (The wheel-sequences $<1,1,1,a>$ will be taken care of by Corollary 7.3 below.) We shall build up most of this reconstructor set by piecing together reconstructor sequences obtained by means of the lemmas in this section.

As we said in Chapter 3, although up to now we have only defined reconstructor sets and reconstructor sequences which contain valency-configurations, we can extend this definition to include other types of configurations. In fact, it is easy to extend the definition so that reconstructor sequences and reconstructor sets may include wheel-sequences. Let W be a wheel-sequence with the property that, for any graph G in $\mathfrak{J}_0$, G is edge-reconstructible if it contains a vertex with wheel-sequence W; we then say that <u>W reconstructs $\mathfrak{J}_0$</u>. A reconstructor set for $\mathfrak{J}_0$ is then defined as a finite set of wheel-sequences each of which reconstructs $\mathfrak{J}_0$. We then obtain a definition of a reconstructor sequence in exactly the same way as that in Chapter 3, except that the term "valency-configuration" is replaced by the term "wheel-sequence".

Before proceeding we need first to introduce some further notation. If G - e has a 3-vertex with wheel-sequence $<a,b,c>$ and e is incident to an (h+2)-face and a (k+2)-face in G, then we say that G - e is of <u>type $<a,b,c;h,k>$</u>.

For any positive integer r, it is easy to determine from $\mathcal{D}'G$ whether or not G has a 4-vertex with wheel-sequence of rim-length r, since G contains such a 4-vertex if and only if some graph in $\mathcal{D}'G$ has a 3-vertex with wheel-sequence of rim-length r. Hence, if for some integer p, G has no 4-vertex with wheel-sequence of rim-length $r \leq p$, and if in $\mathcal{D}'G$ some G - e has a 4-vertex v with wheel-sequence of rim-length $t \leq p$, we then know that the edge missing from G - e is incident to v in G. Let this be the case, let e be incident in G to an (h+2)-face and a (k+2)-face and let v have wheel-sequence <a,b,c,d> in G - e. We then say that G - e is of <u>type <a,b,c,d;h,k></u>.

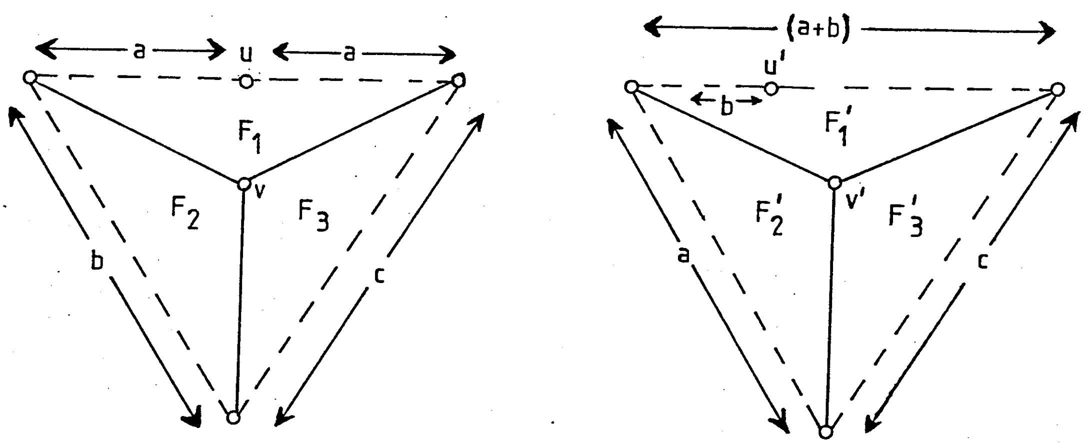
We now have the first of a series of lemmas which will help us to prove Theorem 7.2'.

<u>Lemma 7.1</u>

Let a, b, c be positive integers such that b ≠ 2a ≠ c. Then (<a,a,b,c>,<a,b,a,c>) is a reconstructor sequence for $\mathcal{J}_0$.

<u>Proof</u>

If G has a 4-vertex v with wheel-sequence <a,a,b,c>, then there is a graph G - e of type <2a,b,c;a,a> in $\mathcal{D}'$G (see Figure 7.1(i)).



(i) (ii)

<u>Figure 7.1</u>

Since we know that δG = 4, and since b ≠ 2a ≠ c and e is incident to two (a+2)-faces in G, we can then identify F_1 as the root-face of G - e, so that there is a unique way of reconstructing G from this graph, namely by joining u and v by an edge. Therefore the wheel-sequence <a,a,b,c> reconstructs $\mathcal{J}_0$.

We may therefore assume that G does not have a 4-vertex with wheel-sequence $\langle a,a,b,c\rangle$ (that is, we assume that there is no graph of type $\langle 2a,b,c;a,a\rangle$ in $\mathcal{D}'G$). We may also assume with no loss of generality that $b \geq c$. If G has a vertex v' with wheel-sequence $\langle a,b,a,c\rangle$, then there is a graph $G - e'$ of type $\langle a+b,a,c;a,b\rangle$ in $\mathcal{D}'G$ (see Figure 7.1(ii)). Again, since $b \geq c$, we can identify F_1' as the root-face of $G - e'$; also, since G does not have a 4-vertex with wheel-sequence $\langle a,b,a,c\rangle$, then there is a unique way of reconstructing G from $G - e$, namely by joining u' and v' by an edge. Hence, the wheel-sequence $\langle a,b,a,c\rangle$ reconstructs $\mathfrak{J}_0$, so that $(\langle a,a,b,c\rangle,\langle a,b,a,c\rangle)$ is a reconstructor sequence for $\mathfrak{J}_0$. □

The following lemma is important because it provides the tool which enables us to use the same technique employed in Theorem 3.3 to deal with the valency-configuration R_2.

Lemma 7.2

Assume that G is not edge-reconstructible. Then G has only one edge-reconstruction not isomorphic to it.

Proof

Let H be another edge-reconstruction of G, $H \not\cong G$. Let v be a 4-vertex of G with wheel-sequence $\langle a_0,a_1,a_2,a_3\rangle$ with $a_0 \geq a_i$, $i = 1,2,3$ (see Figure 7.2).

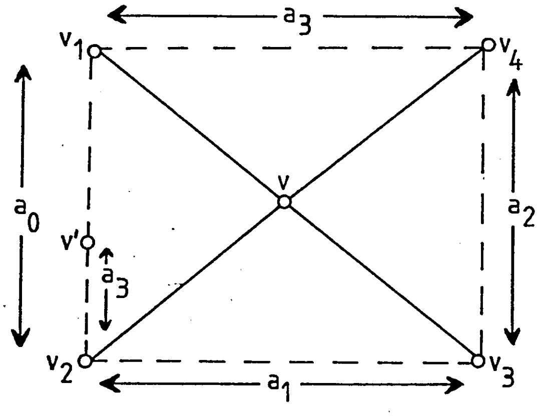


Figure 7.2

Let us consider $G - vv_1$. This is of type $\langle a_0+a_3,a_2,a_1;a_0,a_3\rangle$, so that by the maximality of a_0, we can identify the (a_0+a_3+2)-face incident to v as the root-face; it follows that we can reconstruct from $G - vv_1$ in only two ways: as G or else as $G - vv_1 + vv'$ (see Figure 7.2). But then $H \simeq G - vv_1 + vv'$, and it is the only possible edge-reconstruction of G not isomorphic to G. $\square$

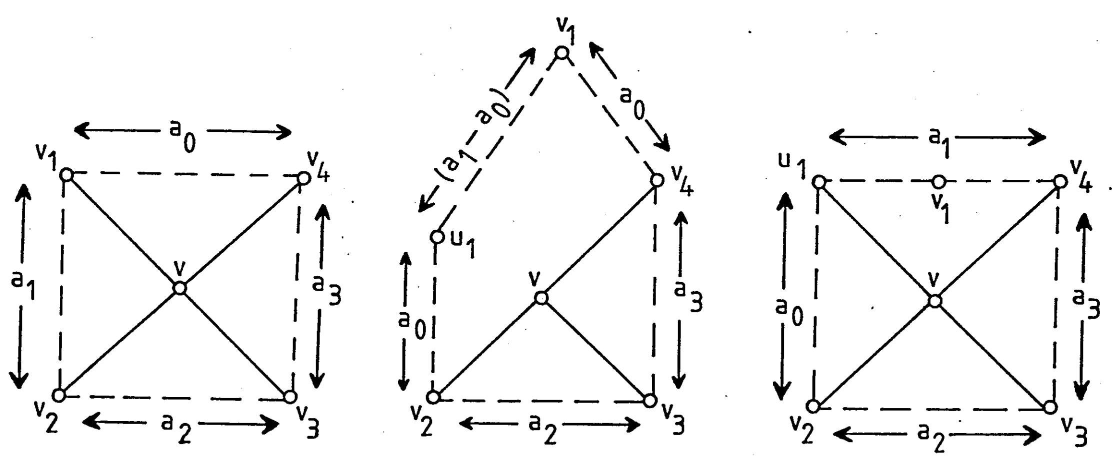
Our last definition now follows. Let v have wheel-sequence $\langle a_0, a_1, \ldots, a_p\rangle$ in G. If for some k in $\{0,1,\ldots,p\}$, $(a_k+a_i) \neq a_j$ for any $\{i,j\} \subset \{0,1,2,\ldots,p\} - \{k\}$, then we say that a_k is <u>special</u> in $\langle a_0, a_1, \ldots, a_p\rangle$.

<u>Lemma 7.3</u>

Let G have a 4-vertex v with wheel-sequence $\langle a_0,a_1,a_2,a_3\rangle$, let the rim-length of this wheel-sequence be r, and let a_0 be special in $\langle a_0,a_1,a_2,a_3\rangle$. If the order of $t = a_1+a_2+a_3$ in Z_r is odd, then G is edge-reconstructible.

<u>Proof</u>

We assume that G is not edge-reconstructible; let H be the other edge-reconstruction of G, $H \neq G$, and let the neighbours of v in G be v_1, v_2, v_3, v_4 as in Figure 7.3(i).



(i) (ii) (iii)

<u>Figure 7.3</u>

Let $W_1(v)$ be the wheel of v in $G - vv_1$, let C_1 be the $W_1(v)$-chain from v_4 to v_2 and let u_1 be the vertex at C_1-distance a_0 from v_2, as shown in Figure 7.3(ii) which illustrates the case $a_1 > a_0$. (We can clearly disregard the case $u_1 = v_1$ which gives rise to an edge-reconstructible graph G by Lemma 7.1.)

Since a_0 is special in $\langle a_0,a_1,a_2,a_3\rangle$, it follows that H and G are associates with respect to $\{G-vv_1, vv_1, vu_1\}$, that is if $G_1 := G - vv_1 + vu_1$, then $H \simeq G_1$ (see Figure 7.2(iii)). We now repeat this process on $G_1 - vv_2$. The wheel of v in $G - vv_2$ is called $W_2(v)$, and the $W_2(v)$-chain from u_1 to v_3 we call C_2. If u_2 is the vertex at C_2-distance a_0 from v_3, then H and G are associates with respect to $\{G_1-vv_2, vv_2, vu_2\}$, that is if $G_2 := G_1 - vv_2 + vu_2$, then $G \simeq G_2$.

This process can be repeated to generate successively graphs $G_1, G_2, G_3, \ldots$ satisfying $G_{2j} \simeq G$ and $G_{2j+1} \simeq H$. (It helps to think of the (a_0+2)-face being "rotated" counter-clockwise about v in successive steps of $a_1, a_2, a_3, a_1, a_2, \ldots$ edges each.)

It is easy to see that $G_{3p} = G$ if p and k are positive integers such that,

$$pt = \underbrace{(a_1+a_2+a_3)+\ldots+(a_1+a_2+a_3)}_{3p \text{ summands}} = rk.$$

Now, by hypothesis, t has odd order in Z_r, so that we can choose p to be odd, and hence $3p$ is also odd. It follows that $H \simeq G_{3p} = G$, a contradiction which establishes the result. □

Corollary 7.2

If G has a 4-vertex with wheel-sequence having odd rim-length, then G is edge-reconstructible.

Proof

Let v have wheel-sequence $\langle a_0,a_1,a_2,a_3\rangle$ with odd rim-length r. We may assume with no loss of generality that $a_0 \geq a_i$, $i = 1,2,3$, so that a_0 is special in $\langle a_0,a_1,a_2,a_3\rangle$. But the order of $a_1+a_2+a_3$ in Z_r divides the order of Z_r, which is r. Therefore the order of $a_1+a_2+a_3$ in Z_r is odd, and hence G is edge-reconstructible by Lemma 7.3. □

When Lemma 7.3 does not work, the following lemma is sometimes helpful.

Lemma 7.4

Let a_0 and a_3 be special in $\langle a_0,a_1,a_2,a_3\rangle$, with $r = a_0+a_1+a_2+a_3$, and $t = a_1+a_2+a_3$. If there exists an odd positive integer q such that,

$$tq + a_2 + a_1 \equiv 0 \bmod r,$$

then $(\langle a_0,a_1,a_2,a_3\rangle,\langle a_0,a_2,a_1,a_3\rangle,\langle a_0,a_1,a_3,a_2\rangle)$ is a reconstructor sequence for $\mathcal{J}_0$.

Proof

Let G have a vertex v with wheel-sequence $\langle a_0,a_1,a_2,a_3\rangle$ and neighbours v_1, v_2, v_3, v_4 as shown in Figure 7.3(i) above. Let us assume that G is not edge-reconstructible, and that H is the other edge-reconstruction of G, $H \not\simeq G$. Let $W_1'(v)$ be the wheel of v in $G - vv_3$, and let C_1' be the $W_1'(v)$-chain from v_4 to v_2. Let u_1' be the vertex at C_1'-distance a_3 from v_2. Then as usual, G and H are associates with respect to $\{G-vv_3, vv_3, vu_1'\}$, that is if $G_1' := G - vv_3 + vu_1'$, then $H \simeq G_1'$.

Now, let $W_2'(v)$ be the wheel of v in $G_1' - vv_2$, let C_2' be the $W_2'(v)$-chain from u_1' to v_1, and let u_2' be the vertex which is at C_2'-distance a_3 from v_4. Again we deduce that if $G_2' := G_1' - vv_2 + vu_2'$, then $G \simeq G_2'$. (Figure 7.4 represents v and its neighbours in G_2'.)

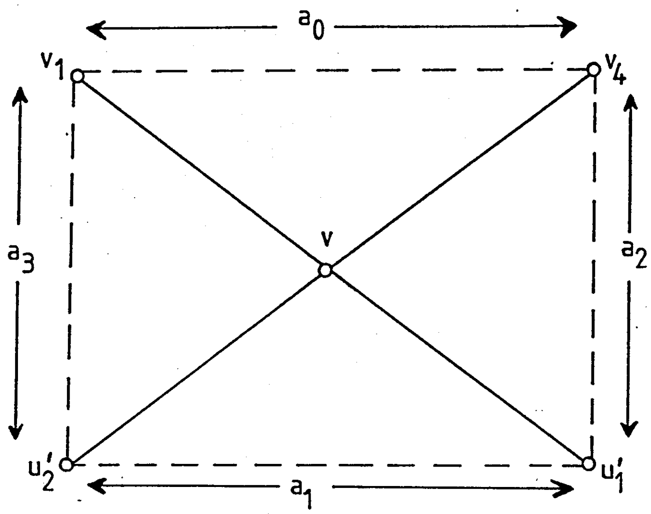


Figure 7.4

Now, let us carry out exactly the same process as in the proof of Lemma 7.3, starting from G with the vertex v adjacent to v_1, v_2, v_3, v_4 (as in Figure 7.3(i)). Again we generate the graphs $G_1, G_2, G_3, \ldots$ satisfying $G_{2j} \simeq G$ and $G_{2j+1} \simeq H$.

Let us consider G_{3q+2}. Since q is odd, then so is $3q+2$. Therefore $G_{3q+2} \simeq H$. But since $a_1 + a_2 + q(a_1 + a_2 + a_3) \equiv 0 \bmod r$, it follows that $G_{3q+2} = G_2'$. Hence $G \simeq H$, a contradiction. Therefore G is edge-reconstructible, so that the wheel-sequence $\langle a_0, a_1, a_2, a_3 \rangle$ reconstructs $\mathcal{J}_0$. In a similar way we can show that $\langle a_0, a_2, a_1, a_3 \rangle$ reconstructs $\mathcal{J}_0$.

Now, let G have a vertex with wheel-sequence $\langle a_0, a_1, a_3, a_2 \rangle$, and let us assume that G is not edge-reconstructible and that H is the other edge-reconstruction of G. In $\mathcal{D}'G$ there is a graph of type $\langle a_0+a_1, a_3, a_2; a_0, a_1 \rangle$. Since a_0 is special in $\langle a_0, a_1, a_2, a_3 \rangle$, then, considering the possible edge-reconstructions from this graph, we obtain that either H or G has a vertex with wheel-sequence $\langle a_1, a_0, a_3, a_2 \rangle$ (that is, $\langle a_0, a_1, a_2, a_3 \rangle$). But since this wheel-sequence reconstructs $\mathcal{J}_0$, and since $H, G \in \mathcal{J}_0$ are not edge-reconstructible, we then obtain a contradiction. Therefore G is edge-reconstructible. and hence $(\langle a_0,a_1,a_2,a_3\rangle,\langle a_0,a_2,a_1,a_3\rangle,\langle a_0,a_1,a_3,a_2\rangle)$ is a reconstructor sequence for $\mathcal{J}_0$. □

The following lemma complements the result of Lemma 7.1.

Lemma 7.5

Let a, b be positive integers. Then $(\langle a,a,2a,b\rangle,\langle a,2a,a,b\rangle)$ is a reconstructor sequence for $\mathcal{J}_0$.

Proof

Let G have a 4-vertex with wheel-sequence $\langle a,a,2a,b\rangle$. If b is not special in $\langle a,a,2a,b\rangle$, then $b = a$, so that 2a is special and $o(a+a+b) = o(3a)$ in Z_{5a} is odd; hence G is edge-reconstructible by Lemma 7.3. If 2a is not special in $\langle a,a,2a,b\rangle$, then $b = 3a$, so that b is special and $o(a+a+2a)$ in Z_{7a} is odd; again we have that G is edge-reconstructible. We may therefore assume that both 2a and b are special in $\langle a,a,2a,b\rangle$. But then G is edge-reconstructible by Lemma 7.4 with $a_0 = 2a$, $a_3 = b$ and $q = 1$. Therefore the wheel-sequence $\langle a,a,2a,b\rangle$ reconstructs $\mathcal{J}_0$.

Now, let G have a 4-vertex with wheel-sequence $\langle a,2a,a,b\rangle$ and let us assume that G is not edge-reconstructible and that H is the other edge-reconstruction of G. Again, we may assume that $b \neq 3a$, otherwise G would be edge-reconstructible by Lemma 7.3. In $\mathcal{D}'G$ there is a graph of type $\langle 3a,a,b;a,2a\rangle$. Since $b \neq 3a$, then, considering the possible edge-reconstructions from this graph we obtain that either H or G has a vertex with wheel-sequence $\langle 2a,a,a,b\rangle$. But since this wheel-sequence reconstructs $\mathcal{J}_0$, and since $H,G \in \mathcal{J}_0$ are not edge-reconstructible, we then have a contradiction. Therefore G is edge-reconstructible, and hence $(\langle a,a,2a,b\rangle,\langle a,2a,a,b\rangle)$ is a reconstructor sequence for $\mathcal{J}_0$. □

As a consequence of Lemmas 7.1 and 7.5 we have the following corollary.

Corollary 7.3

If G contains a 4-vertex with wheel-sequence $\langle a_0,a_1,a_2,a_3\rangle$ such that $a_i = a_j$ for some $i \neq j$, $\{i,j\} \subset \{0,1,2,3\}$, then G is edge-reconstructible. □

SECTION 7.2 - PROOF OF THEOREM 7.2'

As we said above, the proof will consist in exhibiting a reconstructor set for $\mathcal{J}_0$ containing all 5-sequences which have each term less than 4. (We observe that by Corollary 7.3, any wheel-sequence $\langle 1,1,1,a\rangle$, $a \geq 1$, reconstructs $\mathcal{J}_0$.) To obtain this reconstructor set we proceed as follows. First, by means of the lemmas in Section 7.1, we build up a reconstructor set containing all 4-sequences with rim-length less than 16. Hence we may then assume that G has no 4-vertex with such a wheel-sequence, as otherwise it would be edge-reconstructible. Having done this we can then consider 5-sequences with rim-length less than 16, because, as we have already observed, if G has a 5-vertex v with wheel-sequence of rim-length less than 16, and e is an edge incident to v in G, we can then identify the vertex v in $G - e$.

Therefore we shall first show that all 4-sequences with rim-length less than 16 form a reconstructor set for $\mathcal{J}_0$. By Corollary 7.2 we need only consider those 4-sequences with even rim-length, and by Corollary 7.3 we need only consider those 4-sequences in which all four terms are distinct. Since the 4-sequences with distinct terms and the smallest possible rim-length are $\langle a,b,c,d\rangle$ with $\{a,b,c,d\} = \{1,2,3,4\}$, we need only consider 4-sequences with rim-lengths 10, 12 and 14.

To find the number of such 4-sequences we need to know the number of partitions, into exactly four distinct parts, of the integers 10, 12 and 14 respectively. The generating function of the number of such partitions is $\dfrac{x^{10}}{(1-x)(1-x^2)(1-x^3)(1-x^4)}$, (see [L1, p.46]), from which we see that there is one such partition of 10, namely $\{1,2,3,4\}$, two such partitions of 12, $\{1,2,3,6\}$ and $\{1,2,4,5\}$, and five such partitions of 14, these being $\{1,2,3,8\}$, $\{1,2,4,7\}$, $\{1,2,5,6\}$, $\{1,3,4,6\}$ and $\{2,3,4,5\}$. We now proceed to consider the corresponding wheel-sequences.

Considering first the 4-sequences $\langle a,b,c,d\rangle$ with $\{a,b,c,d\} = \{1,2,3,4\}$ we see that 4 is special, so that, since $o(6)$ in Z_{10} is odd, then Lemma 7.3 implies that any such wheel-sequence reconstructs $\mathfrak{J}_0$.

For rim-length 12, we have to consider $(\langle 3,1,2,6\rangle,\langle 3,2,1,6\rangle,\langle 3,1,6,2\rangle)$ and $(\langle 5,1,4,2\rangle,\langle 5,4,1,2\rangle,\langle 5,1,2,4\rangle)$. The first is a reconstructor sequence by Lemma 7.4, with $a_0 = 3$, $a_3 = 6$, $q = 1$, and the second is a reconstructor sequence, also by Lemma 7.4, with $a_0 = 5$, $a_3 = 2$, $q = 1$.

We now consider the 4-sequences with rim-length 14 and distinct terms. As we have seen, these are $\langle a,b,c,d\rangle$ with $\{a,b,c,d\}$ equal to (i) $\{1,2,3,8\}$, (ii) $\{1,2,4,7\}$, (iii) $\{1,2,5,6\}$, (iv) $\{1,3,4,6\}$ and (v) $\{2,3,4,5\}$.

In (i), 8 is special and $o(6)$ in Z_{14} is odd; in (ii) and (v), 4 is special and $o(10)$ in Z_{14} is odd; and in (iii) and (iv), 6 is special and $o(8)$ in Z_{14} is odd. Therefore by Lemma 7.3, all these wheel-sequences reconstruct $\mathfrak{J}_0$.

Having built up our reconstructor set to include all 4-sequences with rim-length less than 16 we can now assume that G has no 4-vertex with such a wheel-sequence, and we turn our attention to the 5-sequences. We could prove lemmas on 5-sequences similar to some of those in Section 7.1. However, since the number of cases to consider here is relatively small and since most of these cases are very straightforward, we prefer to give a case-by-case analysis. Below we give the 5-sequence under consideration in the third column, and in the next column we give the type of G - e from which the graph G is reconstructible if it has a 5-vertex with the wheel-sequence under consideration. In the less straightforward cases we give a fuller proof. (We note here that in some cases in the table below, the order in which the 5-sequences are considered is important, as the proof for some of the wheel-sequences depends on the entries above them. Thus, for example, (<2,2,1,1,1>,<2,1,2,1,1>) is a reconstructor sequence.)

CASE	RIM-LENGTH	WHEEL-SEQUENCES	RECONSTRUCTING TYPE	REMARKS
I	5	<1,1,1,1,1>	<2,1,1,1;1,1>	
II	6	<2,1,1,1,1>		Proof below.
III	7	<3,1,1,1,1>	<3,1,1,2;1,1>	
		<2,2,1,1,1>	<4,1,1,1;2,2>	
		<2,1,2,1,1>	<3,2,1,1;2,1>	By <2,2,1,1,1> above.
IV	8	<2,2,2,1,1>	<4,2,1,1;2,2>	
		<2,2,1,2,1>	<4,1,1,1;2,2>	
		<3,2,1,1,1>		Proof below.
		<3,1,2,1,1>	<3,1,2,2;1,1>	By <3,2,1,1,1> above.
V	9	<3,3,1,1,1>	<3,3,1,2;1,1>	
		<3,1,3,1,1>	<3,1,3,2;1,1>	
		<3,2,2,1,1>	<3,4,1,1;2,2>	
		<3,1,2,2,1>	<3,1,4,1;2,2>	
		<2,3,2,1,1>	<5,2,1,1;3,2>	By <3,2,2,1,1> above.
		<3,2,1,2,1>		Proof below.
		<2,2,2,2,1>	<4,2,2,1;2,2>	

CASE	RIM-LENGTH	WHEEL-SEQUENCES	RECONSTRUCTING TYPE	REMARKS
VI	10	<3,3,2,1,1>	<6,2,1,1 ;3,3>	
		<3,3,1,2,1>	<6,1,2,1;3,3>	
		<3,2,3,1,1>	<5,3,1,1;3,2>	By <3,3,2,1,1> above.
		<3,2,1,3,1>	<3,2,1,4;3,1>	By <3,3,2,1,1> above.
		<3,2,2,2,1>	<3,4,2,1;2,2>	
		<3,2,2,1,2>	<3,4,1,2;2,2>	
		<2,2,2,2,2>	<4,2,2,2;2,2>	
VII	11	<3,3,3,1,1>	<6,3,1,1;3,3>	
		<3,3,1,3,1>	<6,1,3,1;3,3>	
		<1,2,2,3,3>	<1,2,2,6;3,3>	
		<1,2,3,2,3>	<1,2,3,5;2,3>	By <1,2,2,3,3> above.
		<2,1,2,3,3>	<2,1,2,6;3,3>	
		<3,1,3,2,2>	<3,1,3,4;2,2>	
		<2,2,2,2,3>	<4,2,2,3;2,2>	
VIII	12	<3,3,3,2,1>	<6,3,2,1;3,3>	
		<3,3,2,3,1>	<6,2,3,1;3,3>	
		<3,3,2,2,2>	<3,3,2,4;2,2>	
		<3,2,3,2,2>	<3,2,3,4;2,2>	
IX	13	<3,3,3,3,1>	<6,3,3,1;3,3>	
		<3,3,3,2,2>	<6,3,2,2;3,3>	
		<3,3,2,3,2>	<6,2,3,2;3,3>	
X	14	<3,3,3,3,2>	<6,3,3,2;3,3>	
XI	15	<3,3,3,3,3>	<6,3,3,3;3,3>	

<u>Proof for Case II, wheel-sequence <2,1,1,1,1></u>.

Let G have a 5-vertex v with wheel-sequence <2,1,1,1,1> as shown in Figure 7.5, and let us assume that G is not edge-reconstructible

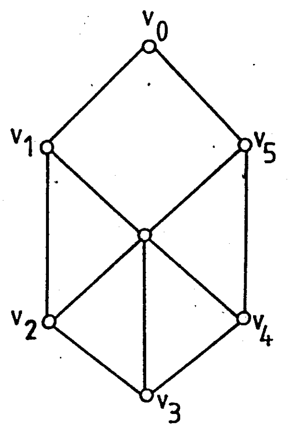


Figure 7.5

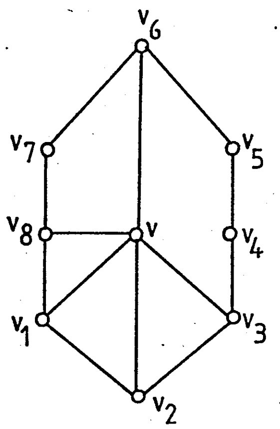


Figure 7.6

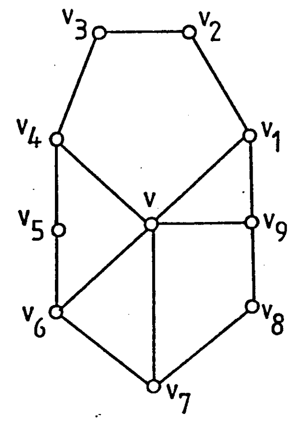


Figure 7.7

and that H is the other edge-reconstruction of G, $H \not\simeq G$. Therefore if $G_1 := G - vv_5 + vv_0$ and $G_2 := G - vv_4 + vv_0$, we have by the usual arguments that $G_1 \simeq H \simeq G_2$. But then, if $G_3 := G_2 - vv_5 + vv_4$, we have that $G_3 \simeq G$. Since $G_3 = G_1$, it follows that $G \simeq H$, a contradiction. □

Proof for Case IV, wheel-sequence <3,2,1,1,1>.

Let G have a 5-vertex v with wheel-sequence <3,2,1,1,1> as shown in Figure 7.6; let us assume that G is not edge-reconstructible and that H is the other edge-reconstruction of G, $H \not\simeq G$. Then if $G' := G - vv_1 + vv_7$, it follows that $G' \simeq H$. Now, since 3 is special in <3,2,1,1,1>, we can apply the same process as in the proof of Lemma 7.3. Thus we obtain $G_1 := G - vv_6 + vv_5$, $G_2 := G_1 - vv_8 + vv_6$, and so on, giving that $G_{2j} \simeq G$ and $G_{2j+1} \simeq H$. But it is easy to check that $G' = G_6$, so that $G \simeq H$, a contradiction. Therefore G is edge-reconstructible. □

Proof for Case V, wheel-sequence <3,2,1,2,1>.

Let G have a 5-vertex v with wheel-sequence <3,2,1,2,1> as shown in Figure 7.7; let us assume that G is not edge-reconstructible and that H is the other edge-reconstruction of G. By the previous entries in the list of 5-sequences we deduce that neither G nor H can have a 5-vertex with wheel-sequence <3,2,2,1,1> or <2,3,2,1,1>. It follows that if $G_1 := G - vv_6 + vv_3$, then $H \simeq G_1$. Also, if $G_2 := G - vv_4 + vv_3$, then $H \simeq G_2$, and therefore if $G_3 := G_2 - vv_6 + vv_4$ then $G \simeq G_3$. But $G_3 = G_1$, and therefore $G \simeq H$, a contradiction. Therefore G is edge-reconstructible. □

This last case completes the proof of Theorem 7.2'.

REMARKS. Parts of the proof of this result are somewhat tedious. Alternative proofs of the Main Theorem of this chapter which do away with most of the case-by-case analysis depend on proving certain results which to date we have been unable to establish. We present one of them as particularly worth attempting.

> Give a reasonably short and direct proof of the following statement: Let G be a 4-connected planar graph and let $G - e$ be such that the edge e is incident to a 4-vertex v in G. Then the wheel-sequence of v in G is reconstructible from $\mathcal{D}'G$.

SECTION 7.3 - EPILOGUE: RECONSTRUCTION FROM EDGE-CONTRACTED SUBGRAPHS

In the previous two sections an important part was played by Theorem 7.3 and especially its corollary, Corollary 7.1. We now present a dual formulation of this result. This will then prompt us to consider briefly another variant of the Reconstruction Problem.

We recall that an edge e of a graph G is said to be contracted if e is deleted and its ends are identified; the resulting graph is denoted by $G.e$. We say that G is contraction-reconstructible if it is uniquely determined, up to isomorphism, from the family $\mathcal{D}''G := \{\{G.e: e \in EG\}\}$, called the contraction-deck of G. Contraction-recognizable classes of graphs are defined in an analogous manner as for the Vertex-reconstruction and Edge-reconstruction

Problems. The two graphs in Figure 7.8 both have the same contraction-deck, so that they are not contraction-reconstructible. Hence, we shall only consider the contraction-reconstruction of graphs with at least four edges.

Figure 7.8

For planar graphs, the Contraction-reconstruction Problem is in some ways a dual of the Edge-reconstruction Problem. In fact, Bondy and Hemminger in [BH1] ask the following question: Let G and H be planar duals; is the edge-reconstruction of G equivalent to the contraction-reconstruction of H ? This is of course true when G is 4-connected, since then every graph G - e has a unique dual (G - e)*, so that if e* is the edge of H corresponding to the edge e of G, then (G - e)* ≃ H.e*; conversely, since G - e is simple and 3-connected, then so is H.e* (see [W3]), and hence G - e is the unique dual of H.e*. One can therefore give a dual of the Main Theorem of this chapter. However, as we said, we are chiefly interested in Theorem 7.3 and Corollary 7.1, particularly because their dual formulations, given as Theorem 7.3* and Corollary 7.1* respectively, apply also to nonplanar graphs.

Theorem 7.3*

Let G be a graph with minimum valency at least 3. Then the valency list of G is reconstructible from $\mathcal{D}''$G. □

Corollary 7.1*

Let G be a graph with minimum valency at least 3, let G.e be any graph in $\mathcal{D}''$G, and let u and v be the two vertices incident to e in G. Then the pair $\{\{\rho u, \rho v\}\}$ is reconstructible from $\mathcal{D}''$G. □

The interesting points to note about these two results are:

(i) Unlike Theorem 7.3 and Corollary 7.1, G here need neither be planar nor 4-connected.

(ii) By the remarks preceding Theorem 7.3* we observe that Theorem 7.3 and Corollary 7.1 follow from Theorem 7.3* and Corollary 7.1* respectively.

(iii) The proof of Theorem 7.3 and its corollary applies to Theorem 7.3* and Corollary 7.1* practically unchanged, apart from the obvious "dual" modifications, like substituting "vertex" for "face" and "edge-contraction" for "edge-deletion". (A root-vertex is defined analogously to a root-face.) In fact it might be easier to visualize the proof of Theorem 7.3 and its corollary by thinking in terms of contractions! The only major difference here is that we have to show first that we can recognize from $\mathcal{D}''$G that G has minimum valency at least 3. This we do in the next lemma. There are also two other slight differences between the proof of Theorem 7.3* and that of Theorem 7.3. Firstly, we note that in Theorem 7.3, we had pairs of *faces* $\{A_i, B_i\}$ such that any two pairs had at least one common face, and we concluded that two such pairs could only have one common face, because G had connectivity greater than 2. In Theorem 7.3*, we obtain analogous pairs of *vertices* $\{u_i, v_i\}$, and again any two such pairs must have at least one common vertex. In this case we obtain that two such pairs can only have one common vertex because G is simple. Secondly, we note that in Theorem 7.3*, all vertices of G might have valency 3. This corresponds to G being maximal planar in Theorem 7.3, a possibility which we did not have to consider there. However it is easily seen that all the vertices of G have valency 3 if and only if for each $G.e \in \mathcal{D}''G$, all vertices of G.e, except one, have valency 3, the exceptional vertex having valency 4. Therefore in this case too, the valency list of G is reconstructible from $\mathcal{D}''G$.

Lemma 7.6

The class of all graphs with minimum valency at least 3 is contraction-recognizable.

Proof

Clearly we can determine from $\mathcal{D}''G$ whether or not G has any 1-vertices, since G has a 1-vertex if and only if some G.e in $\mathcal{D}''G$ has a 1-vertex. If G does not have a 1-vertex, then it is easy to see that $\delta G = d := \text{minimum}\{\{\delta(G.e): G.e \in \mathcal{D}''G\}\}$. We can therefore determine from $\mathcal{D}''G$ whether or not $\delta G \geq 3$. □

In general little work has been done on the Contraction-reconstruction Problem (see [BH1]), and it seems that this problem is no easier than other forms of the Reconstruction Problem. We have already noted above a certain dual relationship between the Edge-reconstruction Problem and the Contraction-reconstruction Problem. There, a result in the Edge Problem about the sizes of faces had a "dual" result in the Contraction Problem about valencies of vertices, with the latter result in fact applying even to nonplanar graphs. Conversely, we believe that just as valencies of vertices often play an important part in edge-reconstruction, so might lengths of circuits play a part in contraction-reconstruction. Thus, for example, just as Eulerian graphs are edge-reconstructible, so might one expect that something can be said about the contraction-reconstruction of bipartite graphs. In fact, we shall now consider this problem. It is interesting to observe that little progress has been done on the vertex-reconstruction or edge-reconstruction of bipartite graphs (see [H4]).

Theorem 7.4

The class of bipartite graphs is contraction-recognizable.

Proof

We shall show that we can determine from $\mathcal{D}''G$ whether or not G has a circuit with an odd number of edges. Clearly, G contains a 3-circuit if and only if some graph in $\mathcal{D}''G$ has multiple edges. We can therefore determine whether or not G has any 3-circuit. If G has a 3-circuit then it is not bipartite. If it does not have a 3-circuit we proceed to determine whether or not it has a 5-circuit. The general argument runs as follows.

Let us assume that it has been determined that G has no $(2p+1)$-circuit for $p = 1,\ldots,k-1$. Let the number of r-circuits in G be c_r and let the total number of r-circuits in all the graphs in $\mathcal{D}''G$ be C_r.

Now, since G contains no $(2k-1)$-circuit, then any $(2k-1)$-circuit of a graph in $\mathcal{D}''G$ must arise through the contraction of some edge which lies on a $2k$-circuit of G. In fact, $C_{2k-1} = 2k \cdot c_{2k}$, so that c_{2k} can be determined.

We now claim that G contains a $(2k+1)$-circuit if and only if there exists a $G.e$ in $\mathcal{D}''G$ which has one of the following properties:

either (i) the number of $2k$-circuits of $G.e$ is more than c_{2k},

or (ii) the number of $2k$-circuits and the number of $(2k-1)$-circuits of $G.e$ are $(c_{2k}-t)$ and r respectively, where $r > t \geq 0$.

First, let us assume that G does contain at least one $(2k+1)$-circuit. Let e be an edge of G which is in a $(2k+1)$-circuit. If e is not in a $2k$-circuit, then the number of $2k$-circuits of $G.e$ is more than c_{2k}, and so (i) holds for $G.e$. Therefore we may assume that the number of $(2k+1)$-circuits and the number of $2k$-circuits containing e are

h and r respectively, with $h > 0$, $r > 0$. Hence the number of $2k$-circuits of $G.e$ is $(c_{2k}+h-r) = c_{2k} - (r-h)$. We may assume that $(r - h) \geq 0$, as otherwise (i) holds for $G.e$. But the number of $(2k-1)$-circuits of $G.e$ is r, and $r > (r - h)$, since $h \geq 1$. Therefore (ii) holds for $G.e$.

Conversely, let us assume that there exists a graph $G.e$ which has one of the properties (i) or (ii). If $G.e$ has property (i), then not all the $2k$-circuits of $G.e$ can be $2k$-circuits of G, so that at least one of them must arise through the contraction of an edge of a $(2k+1)$-circuits of G. If $G.e$ has property (ii), then the number of $2k$-circuits of G containing the edge e is r, since G contains no $(2k-1)$-circuit. Therefore not all the $2k$-circuits of $G.e$ can be $2k$-circuits of G, because otherwise the number of $2k$-circuits of G would be $(c_{2k} - t + r)$, which is impossible since $(c_{2k} - t + r) > c_{2k}$. Therefore we again obtain that G has a $(2k+1)$-circuit. □

<u>Theorem 7.5</u>

Let G be a bipartite graph with minimum valency at least 3, and such that every $G.e$ is 2-connected. Then G is contraction-reconstructible.

<u>Proof</u>

Since the minimum valency of G is at least 3, we can then find a $G.e$ in $\mathcal{D}''G$ with a root-vertex z (we chose $G.e$ and z such that $\Delta(G.e)$ is maximal among all graphs in $\mathcal{D}''G$ and $\rho z = \Delta(G.e)$ in $G.e$). Then $(G.e) - z$ is both connected and bipartite. Since $(G.e) - z$ is connected, we can partition its vertex-set uniquely into two sets V_1 and V_2 such that any edge of $(G.e) - z$ is incident to a vertex from V_1 and a vertex from V_2.

Let the neighbours of z in $G.e$ be $s_1, s_2, \ldots, s_p$ and $t_1, t_2, \ldots, t_q$ such that $s_i \in V_1$ and $t_j \in V_2$ (we note that both p and q are greater than 0). We now claim that the only way to reconstruct from G.e is by adding the edge uv to (G.e) - z and joining u to the vertices s_i only and joining v to the vertices t_j only (or vice-versa). We assume that this is not so and obtain a contradiction. Any other way we reconstruct from G.e we either obtain that u is adjacent to s_i and v is adjacent to s_j, for some $i \neq j$, or else that u is adjacent to t_i and v to t_j. We may assume with no loss of generality that the former case holds. But since (G.e) - z is connected, then there is in (G.e) - z a chain $s_i w_1 w_2 \ldots w_r s_j$ which has an even number of edges, since (G.e) - z is bipartite. Therefore the circuit $us_i w_1 w_2 \ldots w_r s_j vu$ has an odd number of edges, contradicting the fact that G is bipartite. □

Corollary 7.4

Every 3-connected bipartite graph is contraction-reconstructible.

Proof

Let G be a 3-connected bipartite graph. Then any G.e in $\mathcal{D}''G$ is 2-connected. Therefore G is contraction-reconstructible by Theorem 7.5. □

We conclude this epilogue by proving the not too difficult result that maximal planar graphs are contraction-reconstructible. The proof we give further illustrates the duality between edge-reconstruction and contraction-reconstruction. We shall need the following results whose proofs are easy and are omitted.

(1) Let K be a general graph with no loops and whose only multiple edges are two pairs of double edges $\{\{uv,uv\}\}$, $\{\{uw,uw\}\}$, $u,v,w \in VK$. Assume also that if each pair of double edges is replaced by a single edge, then the resulting graph is maximal planar. Then K has a unique plane representation. We shall call such a graph a quasi-maximal-planar graph.

(2) Let G be a maximal planar graph. Then G has an edge uv such that u and v have exactly two common neighbours.

(3) Let G be a maximal planar graph and let $e = uv \in EG$. If u and v have exactly two neighbours in common, then G.e is quasi-maximal-planar.

(4) Let G be a connected graph of order at least 7. Then G is planar if and only if G.e is planar for every edge e of G. (This result is analogous to Theorem 6.1, and can be proved using the characterizations of planar graphs given in Theorems 2.4 and 2.5.)

We now proceed to reconstruct G from $\mathcal{D}''G$ when G is a maximal planar graph. It is easy to see that when G is K_4 it is contraction-reconstructible. We may therefore assume that G has at least five vertices. The first task is to show that we can determine from $\mathcal{D}''G$ whether or not G is maximal planar. Since we can determine the number of edges of G, we may assume that $\varepsilon G = 3 \cdot \nu G - 6$, so that we only have to show that we can recognize whether or not G is planar. If the order of G is at least 7, this recognition is given by (4) above. We therefore have to consider orders 5 and 6. The only nonplanar graph on five vertices is K_5, and since $\varepsilon(K_5) \neq 3 \cdot \nu(K_5) - 6$, we deduce that if G has five vertices it is planar. The only nonplanar graphs with six vertices and $12 = 3 \cdot 6 - 6$ edges can be found in the list of graphs in Appendix I of [H2]. Of these, only the one shown in Figure 7.9 below has the property that all its edge-contracted subgraphs are planar. But this graph has valency list $\{\{3,3,3,5,5,5\}\}$, whereas the only two maximal planar graphs on six vertices have valency lists $\{\{3,3,4,4,5,5\}\}$ and $\{\{4,4,4,4,4,4\}\}$, (again see the list in [H2]). Therefore even this case presents no problem since if G is the graph of Figure 7.9, then by Theorem 7.3* we can tell from $\mathcal{D}''G$ that G is not planar.

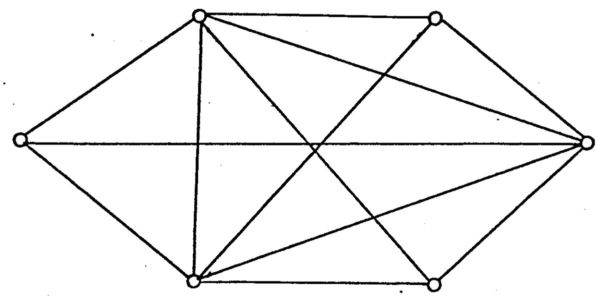


Figure 7.9

Having thus solved the problem of recognition we may now assume that G is maximal planar, and we proceed to reconstruct it from $\mathcal{D}''G$. We shall actually edge-reconstruct $H = G^*$, the dual of G. We note that H is a simple 3-connected graph, so that if we reconstruct H we then obtain G as the unique dual of H. We note also that since H is cubic (that is, every vertex has valency 3), we need only one graph of $\mathcal{D}'H$ to reconstruct H.

But it follows from (1), (2), (3) above that there is a graph G.e in $\mathcal{D}''G$ which has a unique dual. If e^* is the edge of H which corresponds to e, then $(G.e)^* \simeq H - e^*$, so that H can be reconstructed from $H - e^*$.

We therefore have;

Theorem 7.6

Maximal planar graphs are contraction-reconstructible. □

PART IV EXTENSIONS

In this part we are concerned with extending the results and techniques of the previous chapters to the reconstruction of nonplanar graphs. In Chapter 8 we discuss where the previous methods fail, and we indicate where, in the next two chapters, new techniques are needed. In Chapter 9 all graphs which triangulate some surface and have connectivity 3 are shown to be edge-reconstructible. Here, we manage to avoid problems of embeddings by edge-reconstructing two classes of graphs which, between them, constitute a class wider than the class of graphs which triangulate surfaces and have connectivity 3; for these two classes of graphs edge-reconstruction is possible without any consideration of embeddings. In this chapter we also show that graphs which triangulate the torus or the projective plane and have connectivity 3 and minimum valency at least 4 are weakly vertex-reconstructible. In Chapter 10 we show that any graph which triangulates the projective plane is edge-reconstructible. Here, unlike the methods used in Chapter 9 to give the edge-reconstruction of certain graphs which triangulate surfaces, heavy use is made of embedding properties of the graphs under consideration.

CHAPTER 8 EMBEDDINGS OF NONPLANAR GRAPHS

In this final part we shall extend some of our previous results on planar graphs to graphs which triangulate other surfaces. The aim of this chapter is to discuss where our methods for planar graphs now fail, thus motivating our search for new techniques in the next two chapters.

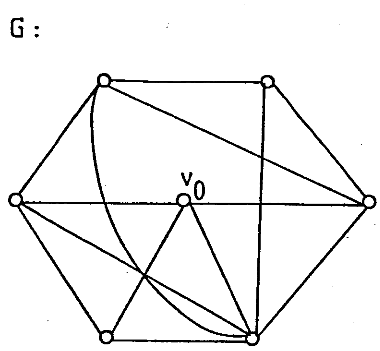
Our previous methods fail primarily for two major reasons, namely that for surfaces other than the plane we lack two fundamental theorems which were crucial in our work: Kuratowski's Theorem (Theorem 2.4) and Theorem 2.7 on the uniqueness of plane representations. We shall first discuss Kuratowski's Theorem.

When trying to reconstruct a nonplanar graph by utilizing its embedding on some surface, the first problem we face is that of recognizing the genus. As can be seen from [F2] and [FM1], recognizing from $\mathcal{D}G$ whether or not G is planar is no straightforward task, although as we saw in Chapter 4, the restriction to maximal planarity does simplify the situation. However, in all of this work, an essential part is played by Kuratowski's Theorem. Since no analogoue of this result is in general available for surfaces other than the plane, these methods will not prove very fruitful for such surfaces. However, although we cannot apply the techniques we used in Chapter 4, we still believe that some of the results there may carry over to nonplanar graphs. As an example we make the following conjecture in which, by a k-representation in a surface S other than the plane we mean an embedding in S, each of whose faces, except one, is a 3-face, the exceptional face being a k-face, $k \geq 4$.

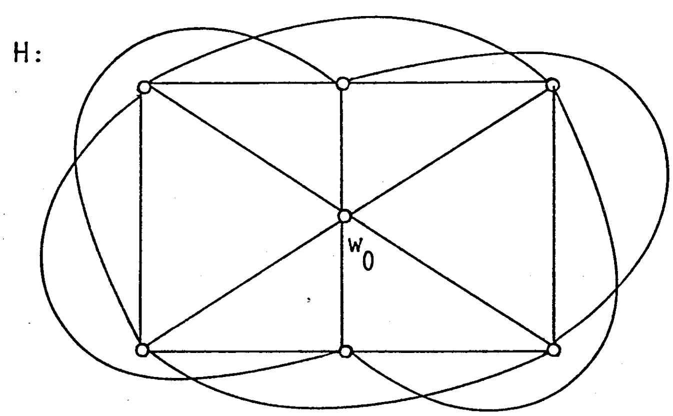
Conjecture

A graph G with minimum valency at least 4 triangulates a surface S if and only if every $G_v \in \mathcal{D}G$ has a ρv-representation in S.

REMARK. If we merely require that $\varepsilon G = 3 \cdot \nu G - 3\chi$ (χ being the Euler characteristic of S) and that each G_v embeds in S, without insisting that each G_v should have a ρv-representation in S, then the conjecture would be false: the graph G in Figure 8.1 (i) is not planar whereas every G_v is planar, and the graph H in Figure 8.1(ii) is not projective but every H_v is projective; moreover, $\varepsilon G = 3 \cdot \nu G - 6$ and $\varepsilon H = 3 \cdot \nu H - 3$. However, $G - v_0$ does not have a ρv_0-representation and $H - w_0$ does not have a ρw_0-representation in the projective plane.



(i)



(ii)

Figure 8.1

Although in general no analogue of Kuratowski's Theorem is known for other surfaces, such a result has recently been found for the projective plane P by Archdeacon [A1] who has shown that for a certain set $I(P)$ (containing 103 graphs!) any graph G is projective if and only if G does not contain a subdivision of any one of the graphs in $I(P)$. (The full list of graphs in $I(P)$

can be found in [GHW1].) It seems quite hopeless, however, to use I(P) to give results on the vertex-recognition of projective graphs. One only has to look at the proofs in [F2], [FM1] and Chapter 4, where only two "forbidden" Kuratowski graphs are involved, to realise what an impossible task it would be to deal similarly with a set of 103 forbidden subgraphs. However, as far as edge-recognition is concerned the problem here, as for the plane, is much more tractable. For the projective plane, the result corresponding to Theorem 6.1 can be formulated as follows:

Theorem 8.1

If G is not a subdivision of any graph in I(P), and has no isolated vertices, then G is projective if and only if each $G - e$ in $\mathcal{D}'G$ is projective.

Proof

Clearly, if some G_e is not projective then neither is G. For the converse, we assume that each G_e is projective but that G is not. Then G contains some subdivision H of some graph in I(P). But since $G \neq H$, and since G has no isolated vertices, then there exists some edge $e_0 \in EG - EH$. Hence $G - e_0$ contains H, and is therefore not projective, giving a contradiction. □

This result will be used in Chapter 10 to give the edge-recognition of graphs which triangulate P. However in Chapter 9, to deal with the edge-recognition of graphs with connectivity 3 and which triangulate surfaces other than the plane or the projective plane, we have to resort to techniques which do not use a Kuratowski-type theorem.

We now consider the question of uniqueness of embeddings. Having recognized the genus of a graph G, the major problem faced when actually trying to reconstruct G arises when the graphs in the vertex-deck, or the edge-deck, do not have a unique embedding on the particular surface on which G is embeddable. (The definition of equivalence of nonplanar embeddings is an obvious extension of the definition of equivalenct plane embeddings given in Chapter 2.) One of the crucial results in our work and in [F1, F2, FM1] which made possible the reconstruction of certain classes of planar graphs was the fact that a 3-connected planar graph has a unique plane representation. Clearly, it would be very desireable if we could obtain an analogue of this result for other surfaces. However, as we shall see, it seems very unlikely that such an analogue can be found.

It is immediately clear that 3-connectedness is not sufficient to guarantee uniqueness of the embedding, as can be seen from the following two embeddings of K_5 in the projective plane. (In Figure 8.2, Greek letters denote points of identification in the projective plane; this same convention is adhered to in similar cases in Part IV.)

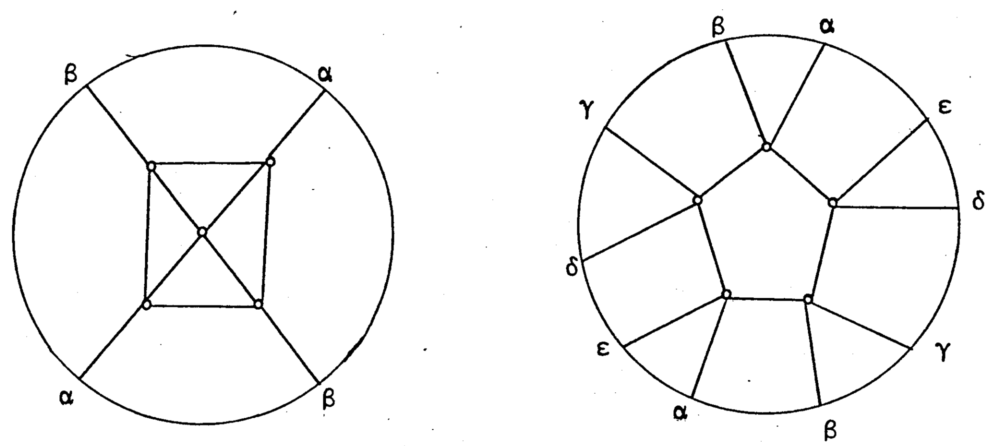


Figure 8.2

As can be seen from examples considered below, this is not an isolated phenomenon, resulting possibly from the fact that K_5 has a small number of vertices. Neither is it due to the fact that P is not orientable, as similar examples on orientable surfaces can be found (see also the discussion of Figure 8.9 below).

One might hope that if G triangulates a surface S, and has sufficiently high connectivity, then any G - e in $\mathcal{D}$'G would have sufficient "rigidity" to ensure that it has a unique representation on S. However, the example shown in Figure 8.3 shows that even this is not true. Here, G is a 5-connected graph which triangulates P. The figure shows a representation of G - ab in P. The other representation is obtained by embedding the edge v_1v_2 inside the face bounded by the circuit av_1bv_2a.

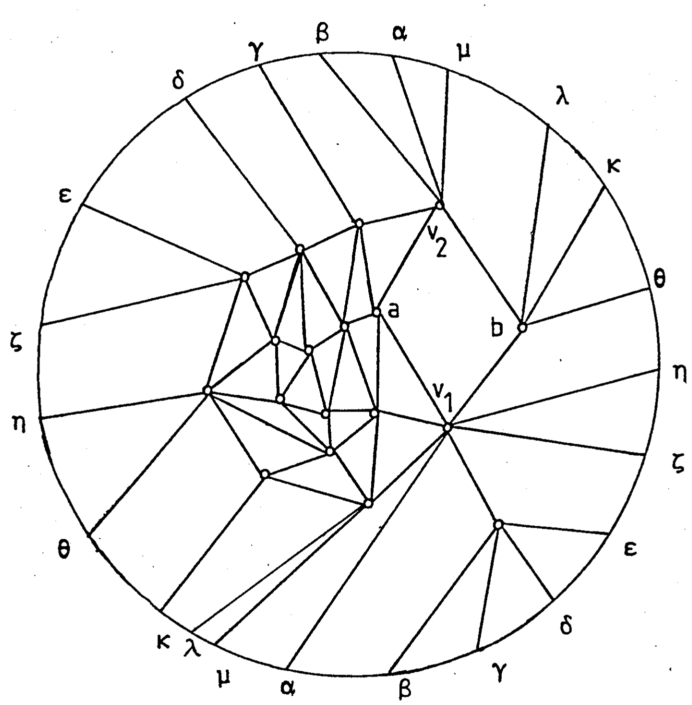
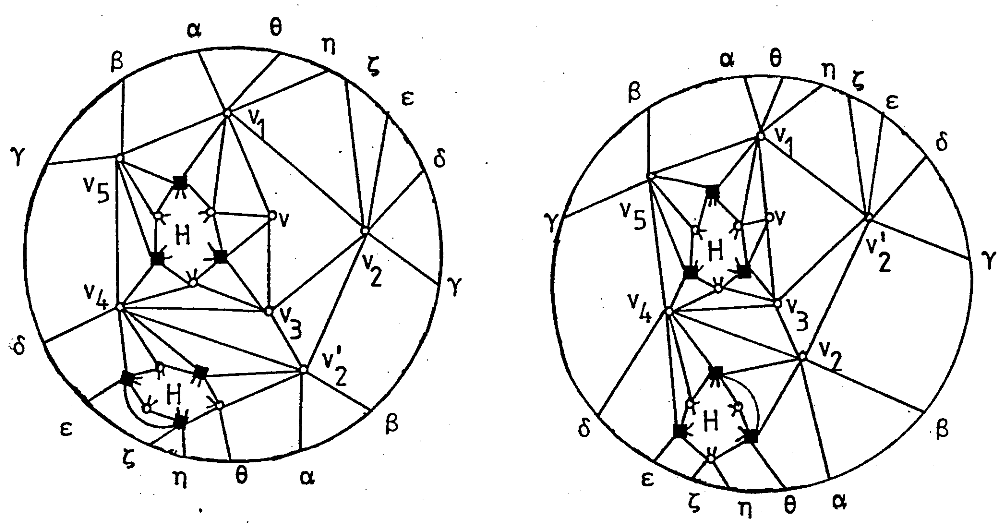
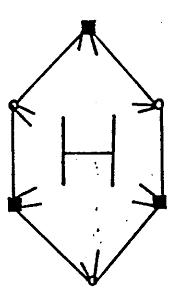


Figure 8.3

However, if we consider a graph G - e (G still triangulates a surface S) such that e is incident in G to a vertex v of minimum valency, we do not even require (for the purpose of reconstruction) that G - e has a unique representation in S. First of all we observe that for reconstruction we need only consider those embeddings in which the vertex v, which we can identify, lies on the circuit bounding the unique 4-face of the embedding, and that moreover, if

$vv_1v_2v_3v$ is such a circuit, then v is not adjacent to v_2; because then G is obtained from one of these representations by joining by an edge the vertex v to the unique vertex , incident to the 4-face, to which v is not already adjacent. All that we require is that such representations are equivalent. However even here we have a counter-example. Figure 8.4 shows two embeddings of $G - vv_2$ in P, where G is 4-connected and triangulates P. We observe also that in $G - vv_2$, the valencies of the two vertices v_2 and v_2' are the same, so that although we do know the valencies of the two vertices to which e is incident in G, this is not sufficient to give unique reconstruction from $G - vv_2$.



Where 

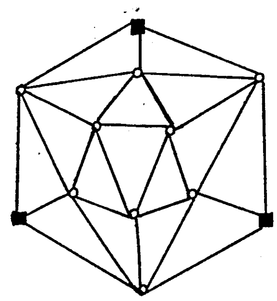
is a schematic representation of: 

Figure 8.4

It seems very unlikely that there could exist graphs G for which this should happen for every $G - e$ with e incident to a vertex of minimum valency in G. However, it does not seem at all evident how to go about proving such results. (If such graphs do exist they would have a rôle analogous to that of collapsible graphs in Chapter 5.)

We conclude this discussion by constructing graphs which provide strong evidence that it is extremely unlikely that any analogue of Theorem 2.7 can be found for surfaces other than the plane. These will be graphs which triangulate some surface S and have high connectivity, but which do not have a unique embedding in S. For a given surface S the construction starts with a graph which, although it does have a unique embedding in S, does not have a face preserving automorphism (with respect to embeddings in S). Thus, let us consider the graph embedded in the projective plane as shown in Figure 8.5.

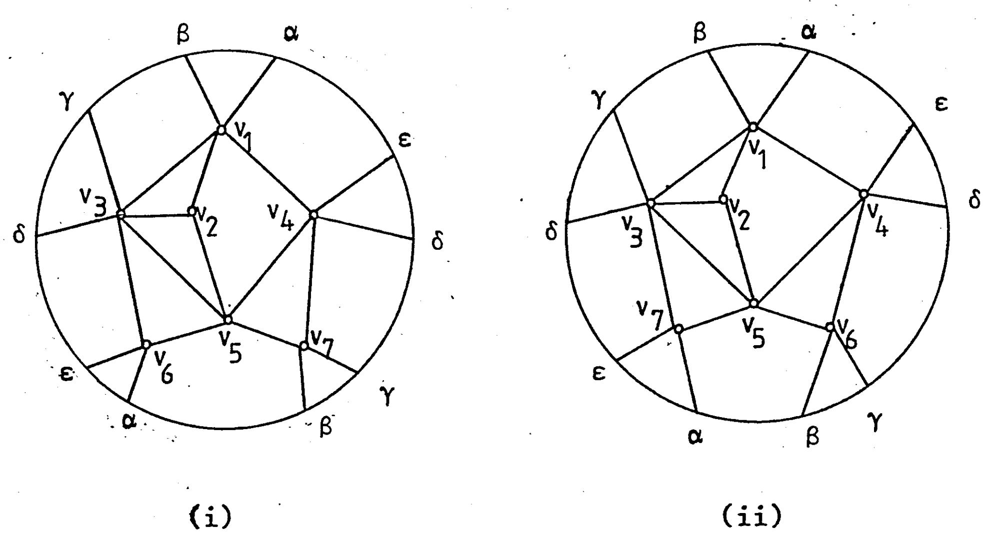
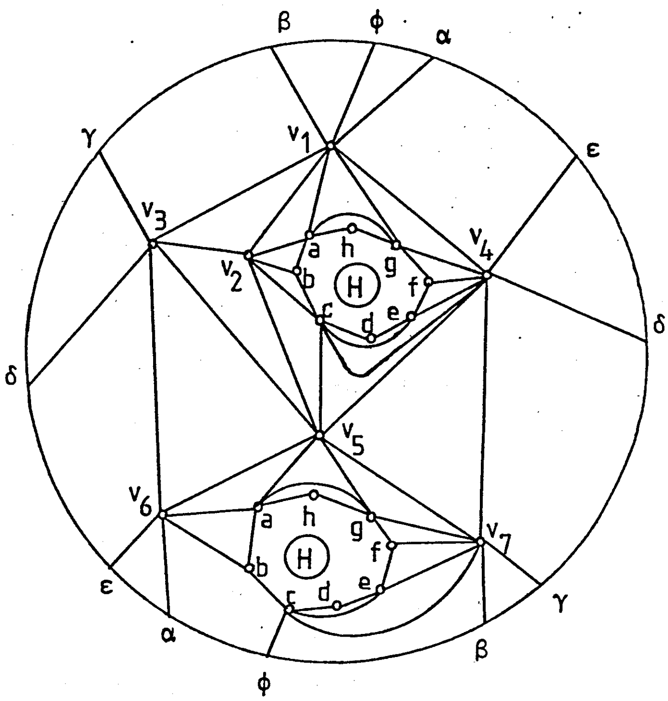


Figure 8.5

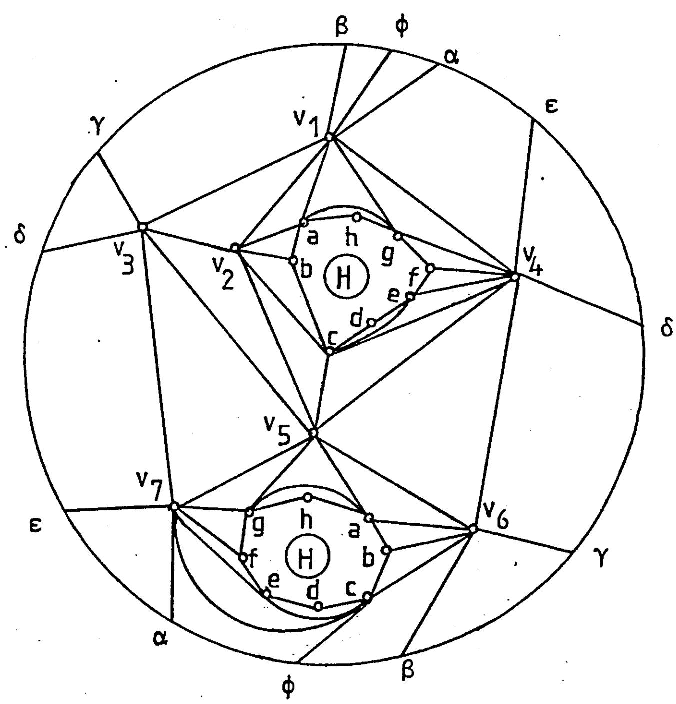
The two embeddings shown in Figure 8.5(i) and 8.5(ii) are clearly equivalent since there exists an automorphism on the graph transposing vertices v_6 and v_7 and leaving every other vertex fixed. But now, starting from this initial "framework" we shall construct two non-equivalent embeddings of a graph which triangulates P. We do this by embedding two plane graphs inside the faces bounded respectively by the circuits $v_1v_2v_4v_1$ and $v_5v_6v_1v_7v_5$, in such a way that the resulting graph triangulates P and that the similarity between the vertices v_6 and v_7 is lost. This is done so that the circuit $v_4v_5v_7v_4$, which bounds a face in the first representation, cannot be replaced in the second representation by any other circuit bounding a face. By this we mean that there is no automorphism ψ on the resulting graph such that the vertex-set $\{\psi v_4, \psi v_5, \psi v_7\}$ induces a circuit which bounds a face in the second representation. Since the circuit $v_4v_5v_7v_4$ does bound a face in the first representation, this would mean that the two representations are not equivalent. In Figure 8.6 we see the result of such a construction. Here, the graphs which were embedded inside the faces bounded by $v_1v_2v_5v_4v_1$ and $v_5v_6v_1v_7v_5$ were chosen so that the resulting graph would have connectivity 4 and minimum valency 5. Under less stringent conditions, simpler graphs could have been chosen. We now proceed to show that in fact the circuit $v_4v_5v_7v_4$ cannot be replaced (in the sense described above) by another circuit bounding a face in R'. The maximum valency of the graph is 9, and the only 9-vertex is v_4; moreover, the only 8-vertices are v_7, v_5, v_1. Now, the valencies of the vertices v_4, v_5, v_7 are 9, 8, 8 respectively, so that the circuit $v_4v_5v_7v_4$ can only be mapped on a circuit v_4xyv_4, where $x,y \in \{v_7, v_5, v_1\}$. However none of these 3-circuits bounds a face in R'. Therefore R and R' are not equivalent.

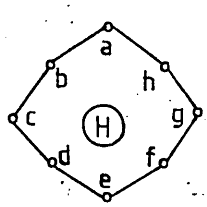
Using this construction other examples can be found. However we could not find, by this or any other method, a 5-connected graph which triangulates P and which has nonequivalent embeddings in P, although we believe that such graphs exist.

R :



R':



Where 

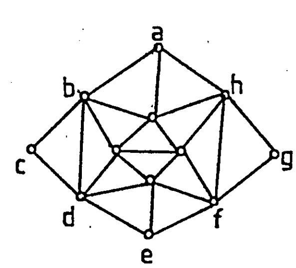
is a schematic representation of: 

Figure 8.6

We can employ a similar construction on the torus. For example, we could start with the initial "framework" shown in Figure 8.7, and embed plane graphs inside the faces bounded by the circuits $v_1v_2v_3v_7v_8v_1$ and $v_4v_5v_6v_7v_3v_4$ in such a way that we end up with nonequivalent embeddings as we did above. We note that in this case, the circuits $v_1v_2v_3v_7v_8v_1$ and $v_4v_5v_6v_7v_3v_4$ are 5-circuits; therefore by judicious choices of the graphs which we embed inside the 5-faces bounded by these circuits, we can obtain a 5-connected graph which triangulates the torus and has nonequivalent embeddings on the torus.

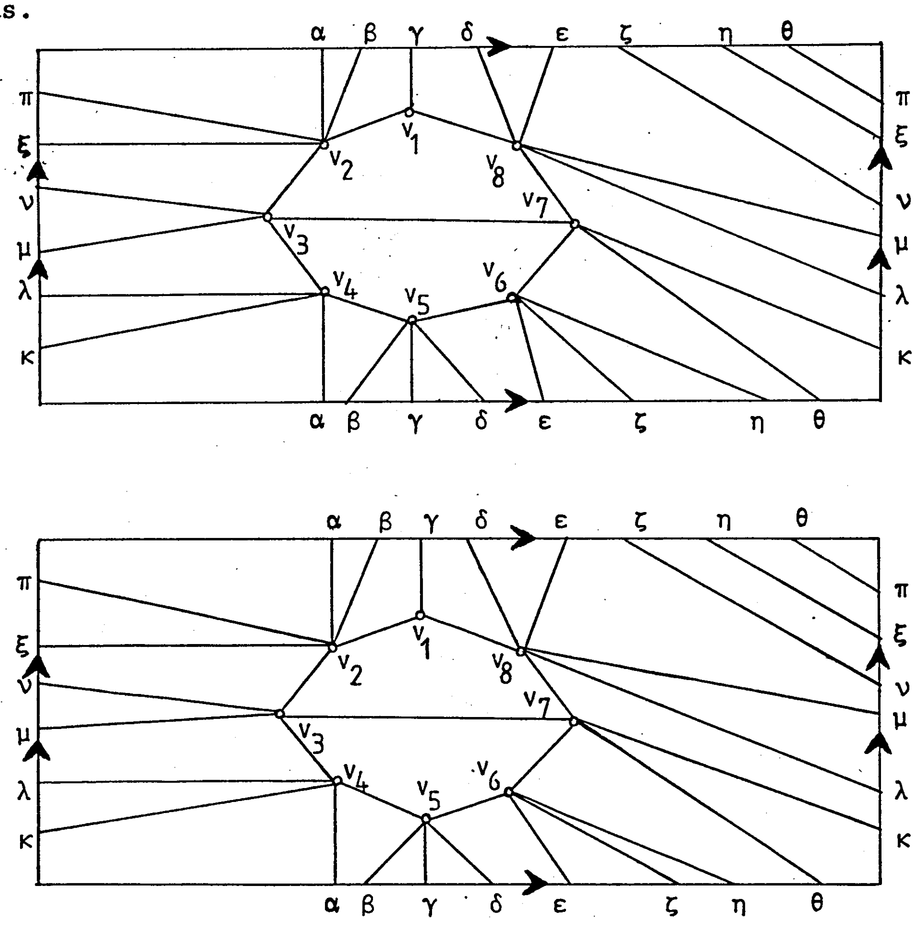


Figure 8.7

Before concluding this discussion on Theorem 2.7 and on the lack of an analogous result for surfaces other than the plane, we note that, when trying to apply our previous methods to reconstruct nonplanar graphs, difficulties are not encountered solely because of non-uniqueness of embeddings. In fact, in the whole of Chapter 5 and in most of Chapter 6 we had to deal with planar graphs which were not 3-connected, and in many cases we had to consider nonequivalent plane representations. The methods used here cannot be applied to nonplanar graphs because the theory of bridges is not sufficient to deal with embeddings on any surface other than the plane. Hence we can no longer say that, for example, one embedding can be changed into another by a sequence of bridges transfers, a result which we specialized to k-representations in Theorem 5.2, the theorem which made possible all the subsequent work of Chapter 5. Neither can we say that if a circuit bounds a face in one representation but not in another, then the graph is not 3-connected. This is why it is no longer true that a 3-connected graph has a unique embedding (see proof of Theorem 2.7). In fact bridges are involved in both the proof of Kuratowski's Theorem (see [BM1] or [O1]) and that of Theorem 2.7, indicating further that the failure of the theory of bridges in dealing with nonplanar embeddings can be regarded as the prime reason why many of our previous techniques cannot be easily applied to nonplanar graphs.

Hence, methods which we used in Chapter 5 and Chapter 6 to deal with planar graphs with nonequivalent representations will not work now. Thus in Chapter 10, where we consider the edge-reconstruction of graphs which triangulate P, we have to employ other properties of the embeddings which do not involve bridges. In Chapter 9 (§§. 9.1, 9.2) we manage to solve both problems of edge-recognition and edge-reconstruction without considerations of embeddings, whereas in Section 9.3, we make uniqueness of embeddings work even for a nonplanar graph G by making use of separating sets of vertices which separate G into two components, one of them being planar, and then invoking results on the uniqueness of plane embeddings for this component.

As a final remark, it is interesting to observe that when studying the reconstruction of maximal planar graphs (and planar graphs in general) the easier results to prove are those for graphs with high connectivity, since then one is assured that at least some graphs in the deck have unique plane representations. In what follows, no such criterion for uniqueness of embeddings is available. Being thus forced to search for new techniques, we obtain in this case stronger results for graphs with connectivity 3 than for graphs with higher connectivity.

CHAPTER 9 *GRAPHS WHICH TRIANGULATE SURFACES AND HAVE CONNECTIVITY 3*

In this chapter we shall be primarily concerned with the edge-reconstruction of graphs which triangulate surfaces and have connectivity 3. As we said in the previous chapter, since we are dealing with graphs which are not necessarily planar or projective we have to solve the problem of recognition without the use of a Kuratowski-type theorem. Moreover, as we have seen, we cannot use uniqueness of embeddings to show reconstruction. We shall solve these problems in Sections 9.1 and 9.2 by edge-reconstructing two classes of graphs which, between them, constitute a class wider than the class of graphs which we actually want to edge-reconstruct.

MAIN THEOREM OF CHAPTER 9

Any graph which triangulates a surface and has connectivity 3 is edge-reconstructible.

In Section 9.3 we then show that graphs which triangulate the torus or the projective plane and have connectivity 3 and minimum valency at least 4 are weakly vertex-reconstructible.

We first have the following result. (We remind the reader that the number of separating r-sets of G is denoted by $s_r G$.)

<u>Lemma 9.1</u>

Let G be a graph with connectivity κ. Then $s_\kappa G$ is reconstructible from $\mathcal{D}G$, and hence from $\mathcal{D}'G$.

<u>Proof</u>

(Since we can, by Theorem 3.2, reconstruct $\mathcal{D}G$ from $\mathcal{D}'G$, for a graph with no isolated vertices, then we need only show that $s_\kappa G$ can be reconstructed from $\mathcal{D}G$.) we shall only consider those graphs

G_v which have connectivity $\kappa - 1$. For any such G_v, the vertex v is in at least one separating κ-set in G. In fact, if $s_{\kappa-1}(G_v) = i$, then v is in exactly i separating κ-sets in G. In this way, we can reconstruct, for every $i > 0$, the number k_i of vertices of G which are in exactly i separating κ-sets. Also, we know k, the total number of vertices which are in at least one separating κ-set of G, since k is equal to the number of graphs G_v which have connectivity $\kappa - 1$. But then we have that,

$$k = \kappa\cdot(s_\kappa G) - \sum_{i\geq 2} (i-1)k_i,$$

from which $s_\kappa G$ can be found. □

We shall also need the following definition in Section 9.3. Let S be a surface and $A \subset S$. Then if there is a set D homeomorphic to the open disk such that $A \subset D \subset S$ we say that A is <u>contractible to zero in S</u>, or simply <u>contractible in S</u> provided there is no ambiguity with the term "contractible" as we have already defined it.

SECTION 9.1 - MINIMUM VALENCY 3: EDGE-RECONSTRUCTION

Let $\mathcal{J}_1$ be the class of graphs which have minimum valency 3, and which have the property that for any 3-vertex v, the neighbours of v induce a 3-circuit. Clearly, any graph which triangulates some surface and has minimum valency 3 is included in $\mathcal{J}_1$. So it is sufficient to show that the class $\mathcal{J}_1$ is edge-reconstructible.

<u>Lemma 9.2</u>

The class $\mathcal{J}_1$ is vertex-recognizable, and hence edge-recognizable.

<u>Proof</u>

This is the same as in Theorem 4.5 where, for a graph G_v, with $\rho_G v = 3$, we used Kelly's Lemma to determine whether or not v is contained in a subgraph of G isomorphic to K_4. □

Theorem 9.1

The class $\mathcal{J}_1$ is edge-reconstructible.

Proof

Let G be a graph in $\mathcal{J}_1$. We may assume that no 3-vertex of G is adjacent to two 4-vertices, as otherwise G would contain a (3,4,4)-triangle, and so would be edge-reconstructible by Theorem 3.4. Therefore G has a 3-vertex v with at least two neighbours a and b of valency greater than 4. We now show that G is uniquely reconstructible from G - ab. Thus, in G - ab the vertex v is a 3-vertex whose neighbours do not induce a 3-circuit. But since we know the valencies of the vertices to which the edge missing from G - ab is incident in G, then we know that these valencies are at least 5, so that the missing edge is not incident to v. Hence, in any reconstruction from G - ab, the vertex v must remain a 3-vertex. But since we know that G is in $\mathcal{J}_1$, then the only way to reconstruct from G - ab is by joining a and b by an edge. □

SECTION 9.2 - MINIMUM VALENCY AT LEAST 4: EDGE-RECONSTRUCTION

Let $\mathcal{J}_2$ be the class of graphs with connectivity 3, with minimum valency at least 4, and such that if $G \in \mathcal{J}_2$, then any separating 3-set of G induces a 3-circuit in G. By Corollary 2.2, any graph which triangulates some surface and has connectivity 3 and minimum valency at least 4 is in the class $\mathcal{J}_2$. It is therefore sufficient to show that the class $\mathcal{J}_2$ is edge-reconstructible.

Lemma 9.3

Let $G \in \mathcal{J}_2$, and let {a,b,c} be a separating set of G such that G - {a,b,c} has a component H with a minimal number of vertices (minimality being here taken over all separating 3-sets of G). If $\overline{H} = \langle \{VH \cup \{a,b,c\}\} \rangle$, then $\overline{H}$ is 4-connected.

Proof

We assume that the lemma is false and derive a contradiction. By our assumption there exists a separating 3-set x,y,z of $\overline{H}$. (We note that since $\delta G > 3$, then $v\overline{H} > 4$.) Obviously, $\{x,y,z\} \neq \{a,b,c\}$. We may therefore assume that $x \notin \{a,b,c\}$.

We first show that $\{x,y,z\}$ cannot be a separating set for G. Let us suppose, for contradiction, that $\{x,y,z\}$ is a separating set for G, and let G_1, G_2 be two different components of $G - \{x,y,z\}$. Then by the minimality of H, neither G_1 nor G_2 is a subgraph of H. (Clearly, neither is it true that for $i = 1$ or 2, $G_i = H$, since $x \in VH$ and $x \notin VG_i$.) Therefore there exist $v \in VG_1$, $w \in VG_2$, such that $v,w \in VG - VH$. Now, there also exists a chain $C(v)$ from v to x, such that all vertices of $C(v)$ except x are in G_1, and similarly there exists a chain $C(w)$ from w to x such that all the vertices of $C(w)$ except x are in G_2. Since $\{a,b,c\}$ is a separating set for G, then either $v \in \{a,b,c\}$ or else v and x are separated in G by $\{a,b,c\}$. Therefore in any case, $VC(v) \cap \{a,b,c\} \neq \emptyset$; similarly $VC(w) \cap \{a,b,c\} \neq \emptyset$.

We may therefore assume $a \in VC(v)$ and $b \in VC(w)$, so that $a \in VG_1$ and $b \in VG_2$. But since a is adjacent to b, this contradicts the fact that G_1 and G_2 are different components of $G - \{x,y,z\}$. We therefore deduce that $\{x,y,z\}$ is a separating set for $\overline{H}$ but not for G.

Let H_1 and H_2 be two different components of $\overline{H} - \{x,y,z\}$, and and let $h_1 \in VH_1$ and $h_2 \in VH_2$. Since $G - \{x,y,z\}$ is connected, then there exists a chain $C = C[h_1,h_2] = h_1t_1\ldots t_sh_2$ in $G - \{x,y,z\}$. Therefore there must be some vertex t_i in C, such that $t_i \notin VH_1$.

Let t_p be the first such vertex in the sequence $h_1, t_1, \ldots, t_s, h_2$. Then $t_{p-1} \in \{a,b,c\}$, say $t_{p-1} = a$. That is, $a \in VH_1$. Similarly we obtain that VH_2 contains one of b or c, say b. But b is adjacent to a, and this contradicts the fact that H_1 and H_2 are distinct components of $\overline{H} - \{x,y,z\}$. □

Lemma 9.4

Let $\kappa G = 3$, and let $\{a,b,c\}$ be a separating set of G such that G_1 is a component of $G - \{a,b,c\}$. If $x,y \in \{a,b,c\}$, then there is a chain $C = C[x,y]$, such that C is not the edge xy and all the internal vertices of C are in G_1.

Proof

We shall prove the result for $\{x,y\} = \{a,b\}$. Let $z \in VG_1$. Since $\kappa G = 3$, it then follows from Theorem 2.2 that there exist three internally disjoint chains $za_1 \ldots a_r a$, $zb_1 \ldots b_s b$, $zc_1 \ldots c_t c$, such that $a_i, b_j, c_k \in VG_1$. The required chain is then $aa_r a_{r-1} \ldots a_1 z b_1 b_2 \ldots b_s b$. □

Lemma 9.5

Let $G \in \mathcal{J}_2$, let $Q = \{a,b,c\}$ be a separating set of G, and let G_1 be a component of $G - Q$. Then at least one pair $\{x,y\}$ of vertices, $x,y \in Q$, is joined by two internally disjoint chains C_1, C_2, such that neither of the chains C_1, C_2 is the edge xy, and all internal vertices of C_1 and C_2 are in VG_1.

Proof

Let us suppose that there are no two such chains joining a and b. This means that if $G_1' = \langle VG_1 \cup \{a,b,c\} \rangle - E\langle Q \rangle$, then in $G_1' - c$ there are no two internally disjoint chains from a to b. Hence by Theorem 2.1, there exists a vertex $p \in V(G_1' - c)$ which separates a and b in $G_1' - c$. Therefore p is a vertex in VG_1, such that any chain from a to b of the type required by the lemma must contain p.

Now, since $\delta G \geq 4$, then there must be a vertex $q \neq p$ such that $q \in VG_1$. Also, since $\kappa G = 3$, then there must be three internally disjoint chains $C[q,a]$, $C[q,b]$, $C[q,c]$, such that the internal vertices of these chains are in VG_1. Hence one of the chains $C[q,a]$ or $C[q,b]$ must contain p. We may assume that $C[q,a]$ contains p, so that $C[q,b]$ does not. Hence, any chain from q to a with internal vertices in VG_1 must contain the vertex p. Therefore in G, the set of vertices $\{p,b,c\}$ separates q and a. Hence $\langle\{p,b,c\}\rangle \simeq K_3$. Also if G_2 is the component of $G - \{p,b,c\}$ which contains q, then $VG_2 \subset VG_1$. Now, by Lemma 9.4 there exists a chain $C[b,c]$ which is not the edge bc, and such that all the internal vertices of $C[b,c]$ are in G_2. Thus, the two chains $C[b,c]$ and bpc are the chains required by the lemma, with $\{x,y\} = \{b,c\}$. □

Lemma 9.6

Let $G \in \mathcal{J}_2$. Then there exists a separating set $\{a,b,c\}$ of G such that $s_3(G - ab) = s_3(G - ab - ac) = s_3G$.

Proof

Let $\{a,b,c\}$ be as in Lemma 9.3 such that $G - \{a,b,c\}$ has a component H with a minimal number of vertices, and let $\overline{H} = \langle VH \cup \{a,b,c\}\rangle$. Also, let G_1 be another component of $G - \{a,b,c\}$. Then by Lemma 9.5, there exist $x,y \in \{a,b,c\}$ which are joined by two internally disjoint chains C_1', C_2' such that none of C_1', C_2' is the edge xy, and all the internal vertices of C_1', C_2' are in VG_1. We may assume without loss of generality that $\{x,y\} = \{a,c\}$.

We first show that $s_3(G - ab) = s_3G$. Let us suppose for contradiction that $s_3(G - ab) \neq s_3G$. Then $s_3(G - ab) > s_3G$, so that there is a separating set $\{x,y,z\}$ of $G - ab$, which is not a separating set of G. Clearly, $a,b \notin \{x,y,z\}$, and furthermore $\{x,y,z\}$ separates a and b in $G - ab$, since $\{x,y,z\}$ is not a separating set in G. However, since $\overline{H}$ is 4-connected, then there are in $\overline{H}$, at least two other internally disjoint chains C_2, C_3 from a to b, apart from the edge ab and the chain $C_1 = acb$. Moreover, by Lemma 9.4, there is another chain C_4 from a to b in G_1. Therefore in $G - ab$ there are four internally disjoint chains C_1, C_2, C_3, C_4 from a to b, contradicting the fact that $\{x,y,z\}$ separates a and b in $G - ab$. Hence $s_3(G - ab) = s_3G$.

We now show that $s_3(G - ab - ac) = s_3(G - ab)$. Again we suppose for contradiction that this is not so. It therefore follows that $s_3(G - ab - ac) > s_3(G - ab)$. Hence there exists a separating set $\{x',y',z'\}$ of $G - ab - ac$ which is not a separating set in $G - ab$. Thus, $a,c \notin \{x',y',z'\}$, and furthermore $\{x',y',z'\}$ separates a and c in $G - ab - ac$, since it is not a separating set for $G - ab$. However, since $\overline{H}$ is 4-connected, there are in $\overline{H}$, two internally disjoint chains C_3', C_4' from a to c, apart from the edge ac and the chain abc. Therefore in $G - ab - ac$ there are four internally disjoint chains C_1', C_2', C_3', C_4' from a to c, contradicting the fact that $\{x',y',z'\}$ separates a and c in $G - ab - ac$. Hence $s_3(G - ab - ac) = s_3(G - ab)$. □

<u>Theorem 9.2</u>

The class J_2 is edge-recognizable.

<u>Proof</u>

We shall establish the theorem by proving the following statement:

Let G have connectivity 3 and minimum valency at least 4 and let $s_3G = k$ (we recall that k is reconstructible from

$\mathcal{D}'G$ by Lemma 9.1). Then $G \in \mathcal{J}_2$ if and only if

(i) there exists some $G - e$ in $\mathcal{D}'G$ with $s_3(G - e) = k$;

(ii) in every such $G - e$, any separating 3-set induces either K_3 or $K_3 - e$; if moreover $\{x,y,z\}$ is a separating 3-set of $G - e$ such that x is not adjacent to y in $G - e$, and if $\{x',y',z'\}$ is any other separating 3-set of $G - e$ not isomorphic to K_3, then $x,y \in \{x',y',z'\}$;

(iii) there exists at least one $G - e_0$ in $\mathcal{D}'G$ with $s_3(G - e_0) = k$, such that $G - e_0$ has a separating 3-set $\{a,b,c\}$ with a not adjacent to b, and with $s_3(G - e_0 - bc) = k$.

If $G \in \mathcal{J}_2$, then (i) and (iii) follow from Lemma 9.6, whereas (ii) follows from the definition of $\mathcal{J}_2$. We therefore have to prove the converse. We assume that (i), (ii) and (iii) hold and consider $G - e_0$. If we suppose that $G \notin \mathcal{J}_2$, then $e_0 \neq ab$. Now since $s_3(G - e_0 - bc) = k$, then $s_3(G - bc) = k$ (otherwise if $s_3(G - bc) \neq s_3(G - e_0 - bc)$, then $s_3(G - bc) < s_3(G - e_0 - bc) = k$, so that $s_3G \leq s_3(G - bc) < k$, which is impossible). But $\{a,b,c\}$ is a separating set of $G - bc$ such that $\{a,b,c\}$ does not induce K_3 or $K_3 - e$ in $G - bc$, contradicting (ii). □

Theorem 9.3

The class $\mathcal{J}_2$ is edge-reconstructible.

Proof

This follows from the proof of Theorem 9.2, since it is evident there that the only way to reconstruct from $G - e_0$ is by joining the two vertices a and b by an edge. □

SECTION 9.3 - MINIMUM VALENCY AT LEAST 4: WEAK VERTEX-RECONSTRUCTION

In this section we shall show that the class of graphs which triangulate the torus or the projective plane and have connectivity 3 and minimum valency at least 4 is weakly vertex-reconstructible. Although such a graph G is not planar, nevertheless we shall be able to use uniqueness of embeddings. We shall do this by identifying a separating 3-set $\{a,b,c\}$ of vertices of G such that this set separates G into two components H_1, H_2 with $\langle VH_1 \cup \{a,b,c\}\rangle$ a 4-connected maximal planar graph. For this maximal planar graph, we then invoke results on the uniqueness of plane representations. Towards this end we first have to prove some results of a topological nature, the main one being the following theorem.

Theorem 9.4

Let G be a graph which triangulates a surface S, and let K be any triangulation of S such that K^1, the 1-skeleton of K, is an embedding of G in S. Let Q be a set of vertices of K^1 whose deletion disconnects K^1 (that is, the corresponding vertices of G also form a separating set for G). Then $\langle Q\rangle$ separates the surface S, and distinct components of $K^1 - Q$ are contained in distinct regions of $S - \langle Q\rangle$.

This result is intuitively obvious, and to prove that $S - \langle Q\rangle$ is disconnected one might informally proceed as follows. Since $K^1 - Q$ is disconnected, and $K^1 - Q$ is the 1-skeleton of the complex $K - \langle Q\rangle$, then $K - \langle Q\rangle$ is also disconnected, and since the the triangulation K is merely one way of representing the surface S, then $S - \langle Q\rangle$ is disconnected. However, the difficulty lies in the fact that $K - \langle Q\rangle$ is not a complex. We therefore have to proceed differently. We shall use a technique which is very useful when dealing with the separation of a surface by a graph embedded on it; we are referring to the use of the second barycentric subdivision of the triangulation K. (For the definition of barycentric subdivision see [G1]. Also, see [Y1, pp.306-307] for a similar use of the second barycentric subdivision in dealing with the separation of surfaces by graphs.) This technique in fact replaces the regions of $K - \langle Q \rangle$ by subcomplexes of the second barycentric subdivision of K. We first define some notation, especially to point out the few instances where our terminology differs from that of [G1].

The i^{th} barycentric subdivision of K is denoted by B_iK. We note that any subcomplex L of K is automatically subdivided into B_iL when K is subdivided into B_iK. We shall only require the first and the second barycentric subdivisions. The second regular neighbourhood of L in K is defined as in [G1, p. 233] (where it is simply called the regular neighbourhood of L in K and is denoted by N), and we denote it by N_2L. In general we define the i^{th} regular neighbourhood of L in K, denoted by N_iL, as the smallest subcomplex of B_iK which contains the set of simplexes

$$\{s \in B_iK\colon s \text{ has at least one vertex in } B_iL\}.$$

We shall only need the first and second regular neighbourhoods. The subcomplex ${}^cL$ of B_2K is the set of simplexes

$$\{s \in B_2K\colon s \text{ has no vertex in } B_2L\}.$$

(${}^cL$ is denoted by V in [G1, p. 233]). For us, the most important fact about ${}^cL$ is that the number of regions into which L divides S is equal to the number of components of ${}^cL$, and in fact, each region of $S - L$ contains precisely one component of ${}^cL$. If Q is a set of vertices of a graph G, then we also denote by Q the subgraph H of G where $VH = Q$ and $EH = \emptyset$.

To prove Theorem 9.4 we require some preliminary lemmas.

Lemma 9.7

Let G be a graph which triangulates the surface S, and let K be a triangulation of S such that G is isomorphic to K^1. Let Q be a set of vertices of K^1 such that Q separates $u, w \in VK^1$. Then $V(B_1\langle Q\rangle)$ separates u and w in $(B_1K)^1$.

Proof

We first observe that since Q separates u and w, then $u, w \notin Q$, and therefore $u, w \notin V(B_1\langle Q\rangle)$.

In the course of this proof, by a barycentric vertex of $(B_1K)^1$ we shall mean a vertex in $V(B_1K)^1 - VK^1$.

We now assume that the lemma is false, and hence that there exists a chain $C = C[u,w]$ in $(B_1K)^1$ such that $VC \cap V(B_1\langle Q\rangle) = \emptyset$.

Let R_u be the set of all the vertices of VK^1 which can be joined to u in K^1 by a chain which does not contain any vertex of Q. Then clearly, $w \notin R_u$, and w cannot be adjacent in K^1 to a vertex of R_u. We note also that if v is a vertex of K^1, such that $v \notin R_u$ and v is adjacent to a vertex of R_u, then $v \in Q$.

Let w' be the vertex adjacent to w in C. Then w' cannot be in R_u. In fact, since no two vertices of K^1 can be adjacent in $(B_1K)^1$, then w' is a barycentric vertex. Also, no non-barycentric neighbour of w' can be in R_u, since otherwise this vertex of R_u would be adjacent to w in K^1. Therefore the chain C certainly does have at least one vertex which is neither (i) a vertex of R_u, nor (ii) a barycentric vertex adjacent in $(B_1K)^1$ to a vertex of R_u. Let p be the first such vertex in C (starting from u), and let p' be the vertex immediately preceding it in C (see Figure 9.1).

C:

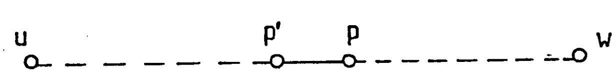


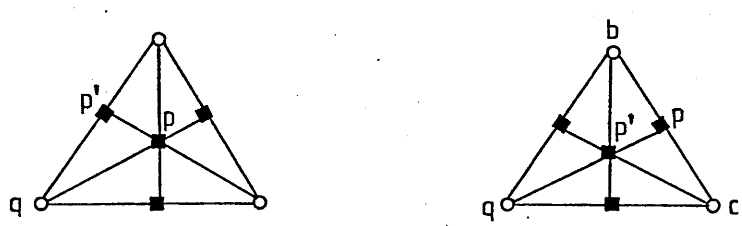
Figure 9.1

Then p' is either a vertex in R_u or a barycentric vertex which is adjacent in $(B_1K)^1$ to a vertex of R_u.

We first observe that p' cannot be in R_u; otherwise p' would be a non-barycentric vertex, since it cannot be a barycentric vertex adjacent to a vertex of R_u. But then, p and p' would be two non-barycentric vertices which are adjacent in $(B_1K)^1$, and this is i impossible. Therefore we may assume that p' is a barycentric vertex adjacent in $(B_1K)^1$ to at least one vertex q of R_u.

We next observe that p must be a barycentric vertex. If we assume the contrary, then p is a vertex of K^1, so that p and q are two vertices of K^1 which are adjacent to the barycentric vertex p' in $(B_1K)^1$; it follows that p and q are adjacent in K^1. But q is in R_u, and therefore either $p \in R_u$ or $p \in Q$, a contradiction (we recall that $VC \cap Q = \emptyset$). We may therefore assume that p is also a barycentric vertex.

Now, if p' is adjacent in $(B_1K)^1$ to only two non-barycentric vertices, then p' is the barycentre of an edge of K^1, and hence q is adjacent to p in $(B_1K)^1$ (see Figure 9.2(i)), a contradiction.



(i) (ii)

Vertices denoted by ■ are barycentric vertices of $(B_1K)^1$

Figure 9.2

We may therefore assume that p' is adjacent to three non-barycentric vertices q, b, c, such that p is adjacent to b and c (Figure 9.2(ii)), so that b and c are not in R_u. But since q

is in R_u and is adjacent to b and c in K^1, then both b and c must be in Q. But then p is the barycentre of the edge $bc \in E\langle Q\rangle$. Therefore $p \in V(B_1\langle Q\rangle)$, which contradicts the fact that $VC \cap VB_1\langle Q\rangle = \emptyset$. This final contradiction concludes the proof of Lemma 9.7. □

Corollary 9.1

Let G, K, Q, u, w be as in Lemma 9.7, and let N_1Q be the first regular neighbourhood of Q in K. Then $V(N_1Q)$ separates u and w in $(B_1K)^1$.

Proof

First of all we note that since $u,w \notin Q$, then $u,w \notin V(N_1Q)$, because if we assume that $u \in V(N_1Q)$, then either there exists a 1-simplex (au) or else a 2-simplex (abu) in B_1K such that $a \in Q$. But then, a and u are vertices of K^1 which are adjacent in $(B_1K)^1$, and this is impossible.

Now, $V(B_1\langle Q\rangle) \subseteq V(N_1Q)$, so that the result follows by Lemma 9.7. □

Lemma 9.8

Let G, K, Q, u, w be as in Lemma 9.7, and let $N_2\langle Q\rangle$ be the second regular neighbourhood of $\langle Q\rangle$ in K. Then $V(N_2\langle Q\rangle)$ separates u and w in $(B_2K)^1$.

Proof

We first observe that since $u,w \notin Q$, then the second regular neighbourhoods of u and w are disjoint from the second regular neighbourhood of $\langle Q\rangle$, so that $u,w \notin V(N_2\langle Q\rangle)$.

Now, let $Q_1 = V(B_1\langle Q\rangle)$. Then by Lemma 9.7, Q_1 separates u and v in $(B_1K)^1$. But then, we can apply Corollary 9.1, with the set Q replaced by Q_1, and K replaced by B_1K. From this we deduce that if N_1Q_1 is the first regular neighbourhood of Q_1 in B_1K, then $V(N_1Q_1)$ separates u and w in $(B_2K)^1$. But $V(N_1Q_1) \subseteq V(N_2\langle Q\rangle)$ (because $V(B_1Q_1) = Q_1 \subseteq V(B_2\langle Q\rangle)$), from which the result follows. □

We are now in a position to give the proof of Theorem 9.4.

Proof of Theorem 9.4

Let the components of $K^1 - Q$ be $K_1, K_2, \ldots, K_r$. We note that since each one of the K_i is disjoint from $\langle Q\rangle$, then the second regular neighbourhood in K of each K_i is disjoint from the second regular neighbourhood of $\langle Q\rangle$. Therefore each one of the N_2K_i, (and so each set VK_i) is included in ${}^c\langle Q\rangle$, and hence each one of the K_i is in $S - \langle Q\rangle$.

Now, let us assume that N_2K_i and N_2K_j, $i \neq j$, are in the same component of ${}^c\langle Q\rangle$. If $u \in VK_i$, $w \in VK_j$, then the vertices u and w are separated by Q in K^1, but are not separated in $(B_2K)^1$ by $N_2\langle Q\rangle$, a contradiction to Lemma 9.8. Therefore the different N_2K_i are in different components of ${}^c\langle Q\rangle$. But each region of $S - \langle Q\rangle$ contains precisely one component of ${}^c\langle Q\rangle$, so that distinct N_2K_i are in distinct regions of $S - \langle Q\rangle$. But N_2K_i is obtained from the graph K_i by subdividing each edge of K_i twice, so that the distinct K_i lie in distinct regions of $S - \langle Q\rangle$. □

REMARK. Although we do not have uniqueness of embeddings, we have still managed to identify, in Theorem 9.4, a property of separating sets which is independent of the embedding. We shall now use this property for reconstruction.

Throughout the rest of this chapter, S will denote the torus or the projective plane. Let G have connectivity 3 and minimum valency at least 4, and assume that G triangulates S. Let {a,b,c} be a separating set of vertices of G. Then by Corollary 2.2,

$C = \langle\{a,b,c\}\rangle$ is a 3-circuit, and by Theorem 9.4, C separates the surface S in any embedding of G in S. However, since S is the torus or the projective plane, it follows that, in any such embedding of G, the circuit C is contractible in S; also, $G - \{a,b,c\}$ has two components G_1 and G_2, such that $\overline{G}_1 = \langle VG_1 \cup \{a,b,c\}\rangle$ is maximal planar and $\overline{G}_2 = \langle VG_2 \cup \{a,b,c\}\rangle$ triangulates S. The component G_1 will be called C_{in} and G_2 will be called C_{out}.

Theorem 9.5

Let $\mathcal{J}$ be the class of graphs which triangulate S, have connectivity 3 and whose minimum valency is at least 4. Then $\mathcal{J}$ is weakly vertex-reconstructible.

(REMARK. We are only proving weak vertex-reconstruction, that is we are assuming that apart from the vertex-deck we are given the extra information that the graph to be reconstructed triangulates S.)

Proof

Let G be a graph in $\mathcal{J}$, and let C be a separating 3-circuit of G such that the number of vertices of C_{in} is minimal among all separating triangles of G. Then $\overline{C}_{in} = \langle VC \cup VC_{in}\rangle$ is 4-connected. † Let $v \in C_{in}$. We shall show that G is uniquely reconstructible from G_v. Let R be any embedding of G_v in S, and let $w \in C_{in}$, $w \neq v$ (such a vertex w exists since the minimum valency of G is at least 4). Since $\overline{C}_{in}$ is 4-connected, there are three internally disjoint chains $C(a) = C[w,a]$, $C(b) = C[w,b]$, and $C(c) = C[w,c]$ in $\overline{C}_{in} - v$.

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† The proof of this is very similar to that of Lemma 9.3. Alternatively one can prove this more easily by noting that $\overline{C}_{in}$ is a maximal planar graph with no separating triangle (by the minimality of C_{in}) and that $\nu(\overline{C}_{in}) > 4$, since $\delta G > 3$.

Let R' be obtained from R by deleting all vertices of $C_{in} - v$ except those of $C(a)$, $C(b)$, $C(c)$, and then contracting these three chains to single edges aw, bw and cw respectively. Let G' be the graph obtained from G by removing C_{in} and adding a vertex y adjacent to a, b, c. Then clearly G' triangulates S, and R' is an embedding of G' in S; moreover, the two components of $G' - \{a,b,c\}$ are C_{out} (the same one as in G) and the single vertex y (which is the vertex w in R'). Therefore by Theorem 9.4 applied to G', it follows that w and C_{out} are in different regions of $S - C$ in the embedding R'. That is, w is in the region of $S - C$ homeomorphic to the open disk. Hence in R, $C_{in} - v$ is the only subgraph of G_v which lies in this region of $S - C$. But since $\overline{C}_{in} - v$ is 3-connected, it has a unique embedding in the plane (which is a ρv-representation). Since we have started with an arbitrary embedding R, then this argument applies to any embedding of G_v in S. Therefore if we reconstruct from G_v by taking any embedding in S and joining v to the vertices incident to the ρv-face, we have unique reconstruction due to the unique plane embedding of $\overline{C}_{in} - v$. $\square$

The vertex-reconstruction of the class $\mathcal{J}$ of Theorem 9.5 is still incomplete because vertex-recognition has not been proved . The conjecture given in Chapter 8, if true, would solve the problem. However, with the aid of Theorem 9.4, we can, in the particular case under discussion, make another conjecture which we believe is more likely to be true than that of Chapter 8. (We note that the necessity of the conditions in the conjecture is clearly true, since, as we have seen in the proof of Theorem 9.5, G has a separating 3-circuit C

such that the maximal planar graph $\overline{C}_{in}$ is 4-connected.)

In the following conjecture, if C is a separating triangle of a graph F, and F_1, F_2 are the components of $F - VC$, we then write $F = \overline{F}_1 \cup \overline{F}_2$.

<u>Conjecture</u>

Let G be a graph with $\delta G \geq 4$ and $\kappa G = 3$. Then G triangulates S if and only if $\mathcal{D}G$ can be partitioned into three pairwise disjoint subfamilies $\mathcal{D}_1G$, $\mathcal{D}_2G$, $\mathcal{D}_3G$, and there exist graphs H and K such that

(i) H triangulates S and K is a 4-connected maximal planar graph;

(ii) for all $G_v \in \mathcal{D}_1G$, $G_v = H \cup K_v$;

(iii) for all $G_w \in \mathcal{D}_2G$, $G_w = H_w \cup K$

(iv) $|\mathcal{D}_3G| = 3$.

CHAPTER 10 *GRAPHS WHICH TRIANGULATE THE PROJECTIVE PLANE: EDGE-RECONSTRUCTION*

In this chapter, by limiting ourselves to graphs which triangulate the projective plane P we complete the main result of Chapter 9.

MAIN THEOREM OF CHAPTER 10

Graphs which triangulate the projective plane are edge-reconstructible.

In view of Chapter 9 we only have to consider 4-connected graphs which triangulate P. Whereas in Chapter 9 (§§. 9.1, 9.2) we were able to avoid considerations of embeddings by working with the classes $\mathfrak{J}_1$ and $\mathfrak{J}_2$, in this chapter heavy use will be made of embedding properties of the graphs under consideration.

The following theorem, which follows from Theorem 8.1, solves the problem of edge-recognition. We may assume that the graph to be reconstructed does not have a pair of adjacent vertices with minimum valency, as otherwise it would be trivially edge-reconstructible.

Theorem 10.1

Let G be a graph with minimum valency at least 3 and such that no two vertices of minimum valency are adjacent, and let $\varepsilon G = 3 \cdot \nu G - 3$. Then G triangulates P if and only if every $G - e$ is projective.

Proof

If G triangulates P, then clearly every $G - e$ is projective. We therefore have to prove the converse. The only graph H in $I(P)$ for which $\varepsilon H = 3 \cdot \nu H - 3$ has minimum valency 5 and does have a pair of adjacent 5-vertices (this graph is labelled A_2 in the list in [GHW1]). This together with the fact that the minimum valency of G is at least 3 implies that G is not a subdivision of any graph in $I(P)$. Therefore by Theorem 8.1, G is projective, and since

$\varepsilon G = 3 \cdot \nu G - 3$, then G triangulates P. $\square$

We now have two sections, depending on the minimum valency of the graph to be reconstructed. We recall that by Euler's inequality, the minimum valency of any projective graph is at most 5.

SECTION 10.1 - MINIMUM VALENCY 4

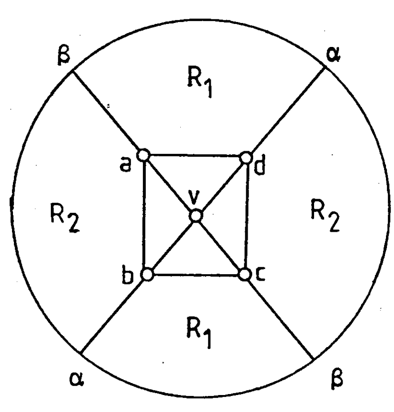
In this section we shall assume that G is a 4-connected graph which triangulates P and has minimum valency 4, and such that no two 4-vertices of G are adjacent.

Lemma 10.1

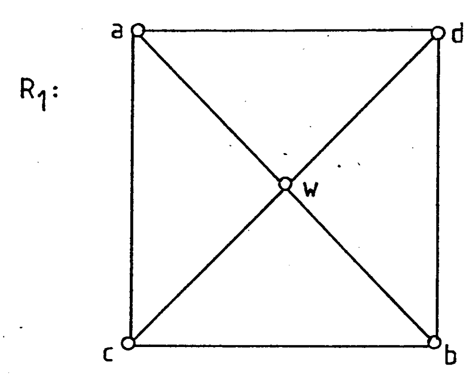
Let v be a 4-vertex of G such that $\langle Nv \rangle \simeq K_4$, and let $w \neq v$ be such that $Nv \subseteq Nw$. Then w is a 4-vertex.

Proof

In any embedding of G in P, the subgraph induced by v and its neighbours must be embedded as shown in Figure 10.1(i), since v is a 4-vertex and is incident only to 3-faces. Then we may assume that w is inside the region R_1.



(i)



(ii)

Figure 10.1

If we assume that $\rho w > 4$, then w must have a neighbour inside one of the regions bounded by the circuits adwa, acwa, cbwc, bdwb (see

Figure 10.1(ii)). This contradicts the fact that G is 4-connected, and shows that w is a 4-vertex. □

Theorem 10.2

Let G have a 4-vertex v such that $\langle Nv \rangle \simeq K_4$. Then G is edge-reconstructible.

Proof

First we observe that the condition on $\langle Nv \rangle$ is recognizable from $\mathcal{D}'G$. In fact, given any $G - e$, we can determine from Kelly's Lemma whether or not the edge e is contained in some subgraph of G isomorphic to K_5. Hence, given any $G - e$, e incident to a 4-vertex w, we have that $\langle Nw \rangle \simeq K_4$ if and only if e is contained in some subgraph of G isomorphic to K_5.

Now, we may assume that v has at most one neighbour of valency 5, as otherwise G would contain a (5,5,4)-triangle and so would be edge-reconstructible by Theorem 3.4. Therefore we may assume that if $Nv = \{a,b,c,d\}$, then $\rho a, \rho b, \rho c \geq 6$.

Now let us consider $G - va$. Since we know that $\langle Nv \rangle \simeq K_4$, and that $\rho a \geq 6$ in G, then in order to have ambiguity in reconstructing G from $G - va$ there must be another vertex $a' \neq a$, such that a' is adjacent to b, c, d, and such that in $G - va$, the valency of a' is is equal to $\rho_G a - 1 \geq 5$. Therefore $\rho_G a' \geq 5$, so that a' cannot be adjacent to a; otherwise , by Lemma 10.1, a' would have valency 4 in G.

Similarly, by considering $G - vb$, we deduce that, in order to have ambiguity in reconstructing G there must be a vertex $b' \neq b$, such that b' is adjacent to a, c, d and not to b. Hence, $b' \neq a'$, since b' is adjacent to a, whereas a' is not. Similarly we may assume that there is a vertex $c' \neq c$, adjacent to a, b, d and not to c, so that $a' \neq c' \neq b'$.

If we now consider $G - vd$, and assume that G is not edge-reconstructible, then there is a vertex $d' \neq d$, adjacent to a,b,c (d' may or may not be adjacent to d). We observe that d' is not equal to either one of a', b', c', since a' is not adjacent to a, b' is not adjacent to b, and c' is not adjacent to c.

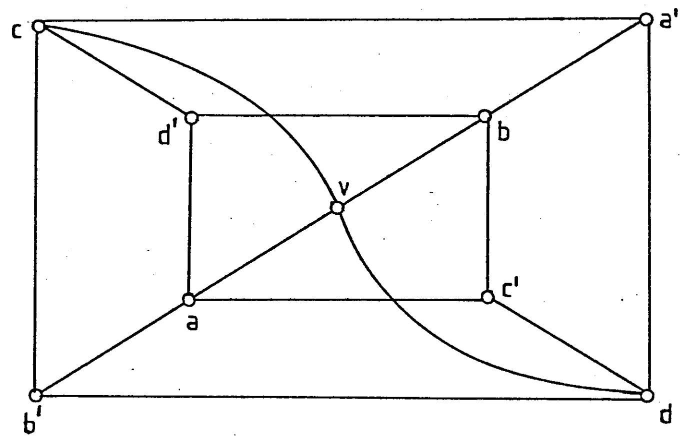


Figure 10.2

But then G contains the subgraph shown in Figure 10.2, and this graph is not projective (since this is graph E_{22} in the list of graphs of $I(P)$ in [GHW1]). This contradiction shows that G is edge-reconstructible. □

In the following we may assume throughout that G contains no 4-vertex v with $\langle Nv\rangle \cong K_4$.

Lemma 10.2

If G is not edge-reconstructible, then every 4-vertex of G is adjacent to two 5-vertices.

Proof

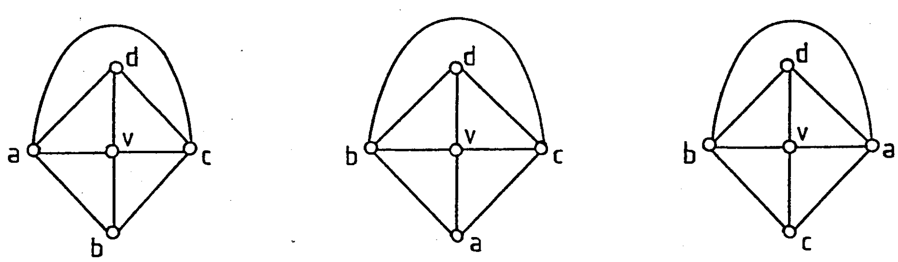
Let v be a 4-vertex of G. We now consider two cases.

Case 1 $\langle Nv\rangle$ is a 4-circuit

Let $Nv = \{a,b,c,d\}$ be such that $E(Nv) = \{ab,bc,cd,da\}$. Let us consider $G - ab$. The subgraph of $G - ab$ induced by the neighbours of the 4-vertex v does not contain a 4-circuit. Since we know that G and any edge-reconstruction of G triangulates P (and so for every vertex w, $\langle Nw \rangle$ is Hamiltonian), we can reconstruct from $G - ab$ either as G or as $G - ab + vp$, for some vertex $p \neq a$. But for $G - ab + vp$ to be an edge-reconstruction of G, we must have that in G, ρa or ρb is 5 (since $\{\{\rho_G a, \rho_G b\}\} = \{\{\rho_H v, \rho_H p\}\}$, where $H = G - ab + vp$). Repeating this argument for each of the edges bc, cd, da we obtain that at least two of a, b, c, d have valency 5 in G. (We note that in fact not more than two vertices from a, b, c, d can have valency 5 in G, as otherwise G would have a (5,5,4)-triangle and so would be edge-reconstructible.)

<u>Case 2</u> <u>$\langle Nv \rangle \simeq K_4 - e$</u>

Let the subgraph of G induced by v and its neighbours be as shown in Figure 10.3(i), and let us assume that the lemma is not true for v, that is, v has at most one 5-vertex as neighbour. We now consider two subcases, obtaining a contradiction in each case.



(i) (ii) (iii)

<u>Figure 10.3</u>

<u>Case 2.1</u> <u>No neighbour of v is a 5-vertex</u>

Let us consider reconstruction from $G - ad$. Since $\rho a, \rho d \geq 6$ in G, then for no $p \in VG$ can we reconstruct as $G - ad + vp$. Hence, v must remain a 4-vertex in any edge-reconstruction of G arising from

$G - ad$. The only way that $\langle Nv \rangle$ can then contain a 4-circuit is either by adding the edge ad or the edge db to $G - ad$; that is, from $G - ad$ we can only reconstruct as G or as $G_1 := G - ad + db$. But then G has only one edge-reconstruction H not isomorphic to it, and $H \simeq G_1$.

We now have a situation similar to those arising in Theorem 3.3 and Lemmas 7.3 and 7.4. We shall use the same technique which we employed in these cases, that is, we shall find successive subgraphs and pairs of edges with respect to which G and H are associates, until we obtain the contradiction that $G \simeq H$.

The subgraph of G_1 induced by v and its neighbours is shown in Figure 10.3(ii). We now repeat the same argument on $G_1 - dc$, and conclude that G and H are associates with respect to $\{G_1 - dc, dc, da\}$, that is, if $G_2 := G_1 - dc + da$, then $G \simeq G_2$. The subgraph of G_2 induced by v and its neighbours is shown in Figure 10.3(iii). We finally apply the same argument to $G_2 - db$, and we deduce that if $G_3 := G_2 - db + dc$, then $H \simeq G_3$. But $G_3 = G$, therefore $H \simeq G$, which is the required contradiction.

<u>Case 2.2</u> <u>The vertex v has exactly one 5-vertex as neighbour</u>

We may assume without loss of generality that either b or c is a 5-vertex in G. If we let b be the 5-vertex, then the same argument used in Case 2.1 is still valid. We may therefore assume that c is the 5-vertex.

By considering $G - ad$, we obtain as before that if $G_1 := G - ad + db$, then $H \simeq G_1$ is the only edge-reconstruction of G not isomorphic to G. Note that in G_1, the valency of a is greater than 5, because otherwise G_1 would contain a (5,5,4)-triangle, and hence would be edge-reconstructible. We now consider $G_1 - ba$. By the same argument, G and H are associates with respect to $\{G_1 - ba, ba, ad\}$, that is, if $G_2' := G_1 - ba + ad$, then $G \simeq G_2'$. (The subgraph of G_2' induced by v and its neighbours is shown in Figure 10.4.)

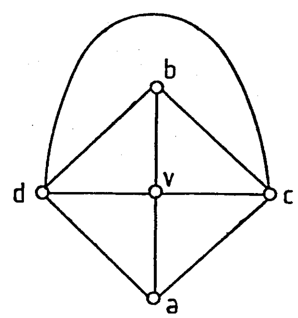


Figure 10.4

If we now consider $G_2' - db$, we similarly obtain that if $G_3' := G_2' - db + ba$, then $H \simeq G_3'$. But $G_3' = G$, so that $H \simeq G$, a contradiction. This final contradiction proves the lemma. □

Theorem 10.3

If G contains no 4-vertex v such that $\langle Nv\rangle \simeq K_4$, then G is edge-reconstructible.

Proof

Let v be a 4-vertex of G. Then by Lemma 10.2, if $Nv = \{a,b,c,d\}$, we may assume that $\rho a = \rho c = 5$ in G. We may therefore assume that a is not adjacent to c in G, as otherwise G would contain a (5,5,4)-triangle. Since $\langle Nv\rangle$ must contain a 4-circuit, then a is adjacent to b and d; similarly c must be adjacent to b and d (see Figure 10.5).

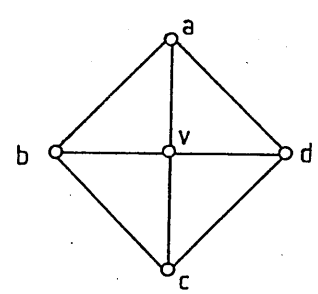


Figure 10.5

Now, since $\rho a = 5$, then $Na = \{b,v,d,x,y\}$ for some vertices $x,y \notin \{b,v,d,c\}$. But $\langle Na \rangle$ contains a 5-circuit, and since v has valency 2 in $\langle Na \rangle$, then x is adjacent to y. We may therefore assume that at most one of x or y is a 5-vertex; otherwise G would contain the valency-configuration S_4 of Theorem 3.4, with $\delta = 4$. Also, we may assume that $\rho x, \rho y > 4$ in G, as otherwise the 5-vertex a would be adjacent to two 4-vertices (v and one of x or y), and so G would contain the valency configuration R_2 of Theorem 3.3, with $\delta = 4$.

We now claim that G is edge-reconstructible. For a contradiction we suppose that it is not and we consider the possible edge-reconstructions from $G - va$. Since the minimum valency of G and of any edge-reconstruction H of G is 4, we deduce that if H is not isomorphic to G, then $H \simeq G - va + vp$, for some vertex $p \neq a$. But in $G - va + vp$, the vertex a is a 4-vertex with at most one neighbour of valency 5. It follows from Lemma 10.2 that H is edge-reconstructible. But G is an edge-reconstruction of H, $G \ncong H$. We have therefore obtained the required contradiction. □

SECTION 10.2 - MINIMUM VALENCY 5

In this section we shall assume that G is a 4-connected graph which triangulates P, has minimum valency 5, and such that no two 5-vertices of G are adjacent. We shall need the following lemma on planar graphs and its corollary.

<u>Lemma 10.3</u>

Let H be a graph which has a k-representation R, and let C be the k-circuit bounding the k-face in R. For any vertex $v \in VH - VC$, let $\rho v \geq 5$, and let there be at most five vertices of C with valency 3. If there are no chords of C in EH, then at least one vertex in $VH - VC$ has valency 5.

Proof

Since there are no chords of C, then no vertex of C has valency 2: otherwise, if $v_0 \in VC$ has valency 2, and v_1, v_{k-1} are the two vertices of C adjacent to v_0, then v_1 is adjacent to v_{k-1}. Therefore any vertex of C has valency at least 3.

Now, we may assume that the k-face F of R is not the unbounded face. Inside the face F we can embed another copy of R, this time with k-face as the unbounded face, so that if we identify the corresponding edges of C, we obtain a maximal planar graph H'. (We note that there can be no multiple edges in H' since H contains no chord of C.)

Clearly, if $v \in VC$, then $\rho_{H'}v = 2\rho_H v - 2$, and if $w \in VH - VC$, then $\rho_{H'}w = \rho_H w$. Therefore H' has at most five 4-vertices, all the other vertices having valency at least 5. Moreover, if ν_k is the number of k-vertices of H', then $\nu_5 = 2x$, where x is the number of 5-vertices of H in $VH - VC$. But Euler's inequality for H' gives

$$2 \cdot \nu_4 + \nu_5 \geq 12,$$

and since $\nu_4 \leq 5$, then $\nu_5 \geq 2$, so that $x \geq 1$, as required. □

Corollary 10.1

Let H have a 6-representation R, and let C be the 6-circuit bounding the 6-face in R. Assume that for any $v \in VH - VC$, $\rho v \gtrsim 5$, and that there are no chords of C in EH. If the order of H is at least 8, then at least one vertex in $VH - VC$ has valency 5.

Proof

Again, since there are no chords of C, any vertex of C has valency at least 3. Therefore to apply Lemma 10.3 to H, all we need to do is to show that not all the vertices of C can have valency 3. To see this, we assume for contradiction that all the vertices of C do in fact have valency 3. Let $VC = \{v_0,v_1,v_2,v_3,v_4,v_5\}$ be such that v_i is adjacent to v_{i+1} (modulo 6), and let v_0' be the other neighbour of v_0 in H. Then v_0 is adjacent to v_4 and v_1. Therefore v_0' is also the other neighbour of v_1 in H, and again we obtain that v_0' is adjacent to v_2. Continuing in this way, we obtain that all the vertices of C are adjacent to v_0'. Now H has at least 8 vertices, so that there must be at least one vertex inside the region bounded by the circuit $v_0'v_iv_{i+1}v_0'$, for some $0 \le i \le 4$. It follows that v_i and v_{i+1} have valency greater than 3 in H. □

A proof of Lemma 10.4 can be found in [GHW1].

<u>Lemma 10.4</u>

The projective plane contains no pair of disjoint non-contractible circuits. □

<u>Lemma 10.5</u>

Let G be a 4-connected graph with minimum valency 5 and such that G triangulates P. If G has a 5-vertex w, such that $\langle Nw\rangle$ is a 5-circuit, then G is edge-reconstructible.

<u>Proof</u>

Let $Nw = \{w_0,w_1,w_2,w_3,w_4\}$ be such that w_i is adjacent to w_{i+1} (modulo 5). Then in $G - w_0w_1$, the vertex w is a 5-vertex such that its neighbours do not induce a Hamiltonian graph. If we assume that G is not edge-reconstructible, and let H be an edge-reconstruction of G not isomorphic to G, then $H \simeq G - w_0w_1 + xy$, where $xy \ne w_0w_1$. By Theorem 10.1, H triangulates P, so that in H, $\langle Nw\rangle$ must contain a $\rho_H w$-circuit. Hence, since $xy \ne w_0w_1$, we deduce that $w \in \{x,y\}$; say $w = x$. But since

$$\{\{\rho_G w_0,\rho_G w_1\}\} = \{\{\rho_H x,\rho_H y\}\} = \{\{\rho_H w,\rho_H y\}\},$$

and since $\rho_H w = 6$, we obtain that one of w_0, w_1 has valency 6 in G.

Similarly we obtain that for all $0 \le i \le 4$, one of w_i, w_{i+1} has valency 6 in G. But then, G has a (6,6,5)-triangle, from which it follows that G is edge-reconstructible, by Theorem 3.4. □

Corollary 10.2

Let G be a 4-connected graph with minimum valency 5 and which triangulates P, and let G be not edge-reconstructible. Then, in a any embedding of G in P, every 5-vertex is on a 3-circuit which is non-contractible in P.

Proof

Let R be an embedding of G in P, and let v be a 5-vertex of G. In R, v is incident to five 3-faces bounded by the circuits $vv_iv_{i+1}v$, $0 \le i \le 4$ (modulo 5). By Lemma 10.5, and since G is not edge-reconstructible, EG must contain a chord of the circuit $v_0v_1v_2v_3v_4v_0$. We may therefore assume, with no loss of generality, that v_0 is adjacent to v_2. But then, the circuit vv_0v_2v cannot be contractible in P, because otherwise the vertices v, v_0, v_2 would constitute a separating 3-set for G, contradicting the fact that G is 4-connected. □

Theorem 10.4

Let G be a 4-connected graph with minimum valency 5 and which triangulates P. Then G is edge-reconstructible.

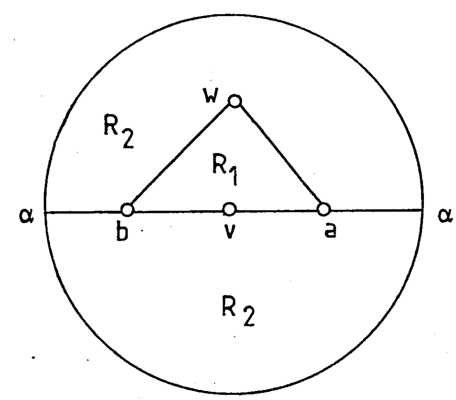
Proof

We assume that G is not edge-reconstructible and obtain a contradiction. Since G is not edge-reconstructible, then no two 5-vertices of G are adjacent, and moreover, G satisfies the conclusion of Corollary 10.2. Let R be an embedding of G in P, and let v be a 5-vertex of G. Then there is a 3-circuit C(v) = avba of R which is non-contractible in P. Hence, if w is another 5-vertex of G, such that in R, w is on a 3-circuit C(w) which is non-contractible in P, it follows from Lemma 10.4 that $C(w) \cap C(v) \neq \emptyset$. We now consider two cases.

<u>Case 1</u> <u>There exists a 5-vertex w ($\neq v$) such that w is adjacent to a and b, and the circuit $awba$ of R is non-contractible in P</u>

Let $W = \{w \in VG$: w is a 5-vertex in G and the circuit $awba$ is non-contractible in $P\}$.

We consider any $w \in W$ and note that, since $awba$ and $avba$ are both non-contractible circuits, then the circuit $avbwa$ is contractible (see Figure 10.6). This assertion is easily proved (see below) by considering $\Pi_1(P,a)$, the fundamental group of P at a, and using the fact that $\Pi_1(P,a)$ is a cyclic group of order 2.



<u>Figure 10.6</u>

Thus the circuit $avbwa$ separates P into two disjoint regions R_1 and R_2, such that R_1 is homeomorphic to an open disk. Let $E(R_1)$ be the set of edges of R which are embedded in R_1, and let $H(w) = \langle E(R_1) \cup \{av, vb, bw, wa\} \rangle$. Then $H(w)$ has a 4-representation in the plane. Let $w_0 \in W$ be such that $\nu H(w_0) \leq \nu H(w)$, for all $w \in W$. We can apply Lemma 10.3 to $H(w_0)$, with the circuit $avbw_0a$ bounding the 4-face of the 4-representation of $H(w_0)$. Since $H(w_0)$ clearly contains no chords of the circuit $avbw_0a$, we deduce that there is a vertex $w' \in VH(w_0) - \{a,v,b,w_0\}$ which has valency 5 in G. Hence any non-contractible circuit $xw'yx$ must be $aw'ba$ (since w' cannot be adjacent to w_0 or v). We conclude that $w' \in W$, and $\nu H(w') < \nu H(W_0)$, a contradiction.

Case 2 There exists no 5-vertex w ($\neq v$) such that w is adjacent to a and b, and the circuit $awba$ of R is non-contractible in P

We may assume that if $w \neq v$ is a 5-vertex, then for some vertex $c \neq b$, the circuit $cwac$ is non-contractible in P. (We may also assume that if v is adjacent to c, then the circuit $avca$ is contractible, because otherwise we revert to Case 1, with the circuit $avca$ replacing the circuit $avba$.)

We first show that it is not possible that every 5-vertex of G is adjacent to the vertex a. Let us suppose the contrary. Since G is not edge-reconstructible, we then have by the reconstructor sequence of Theorem 3.3 that the valency of a in G is at least $5 + \nu_5$. Hence the maximum valency Δ of G is at least $5 + \nu_5$. But Euler's inequality (which is an equality in this case, since G triangulates P) then gives,

$$\nu_5 = 6 + \sum_{k=7}^{\Delta} (k-6)\nu_k > \nu_5,$$

a contradiction.

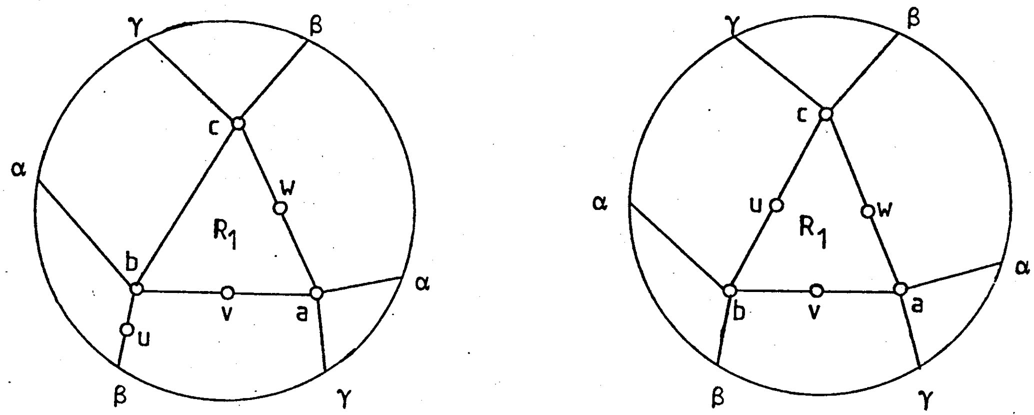
Now, let u be a 5-vertex of G such that u is not adjacent to a, and let $xuyx$ be a non-contractible 3-circuit. Then by Lemma 10.4, and since no two 5-vertices of G are adjacent, we have that

$$\{x,y\} \cap \{b,a\} \neq \emptyset \neq \{x,y\} \cap \{c,a\}.$$

But since u is not adjacent to a, then we must have that

$$\{x,y\} = \{c,b\}.$$

We note now that one of the circuits $bvawcub$ or $bvawcb$ is contractible in P (see Figure 10.7). This assertion is again easily proved (see below) by considering the fundamental group of P at b.



bvawcb is contractible (i)

bvawcub is contractible (ii)

Figure 10.7

We accordingly consider two cases.

Case 2.1 The circuit bvawcb is contractible.

In this case, the circuit bvawcb separates P into two disjoint regions R_1, R_2, where R_1 is homeomorphic to an open disk. Let $E(R_1)$ be the set of edges of R which are embedded in R_1, and let $H = \langle E(R_1) \cup \{bv,va,aw,wc,cb\} \rangle$. Then H has a 5-representation in the plane. We can therefore apply Lemma 10.3 to H, with the circuit bvawcb bounding the 5-face of H (clearly H has no chords of the circuit bvawcb). This implies that there is a vertex $w' \in VH - \{b,v,a,w,c\}$ which has valency 5 in G. Let $pw'qp$ be a non-contractible 3-circuit. Then clearly, $\{p,q\} = \{a,c\}$, since $\{p,q\} \neq \{a,b\}$. But then we revert to Case 1, with the circuit avba replaced by the circuit awca.

Case 2.2 The circuit bvawcub is contractible

Again, the circuit bvawcub separates P into two regions, R_1 and R_2, with R_1 homeomorphic to an open disk. Let $E(R_1)$ be the set of edges of R which are embedded in R_1, and let $H = \langle E(R_1) \cup \{bv,va,aw,wc,cu,ub\}\rangle$. We may assume that $VH \geq 8$, because otherwise H would have a 6-vertex z adjacent to the vertices a, b, c, u, v, w, and therefore z would be a 6-vertex of G, adjacent to three 5-vertices, from which we deduce that G is edge-reconstructible by Theorem 3.3. It is also clear that H has no chords of the circuit bvawcub. We may therefore apply Corollary 10.1 to H, with the circuit bvawcub bounding the 6-face of H. We thus infer that there is a vertex $w' \in VH - \{a,b,c,u,v,w\}$ which has valency 5 in G. Let pw'qp be a non-contractible 3-circuit. Then since $\{p,q\} \neq \{a,b\}$, one of the following must hold: either $\{p,q\} = \{a,c\}$ or $\{p,q\} = \{c,b\}$; we may assume with no loss of generality that $\{p,q\} = \{a,c\}$. But then we revert to Case 1, with the circuit avba replaced by the circuit awca. □

Verification of assertions in the proof of Theorem 10.4

We require a few preliminary results. All these are of an elementary nature and can be found in [P1]. The definitions of paths, their products, homotopy and fundamental group can likewise be found in [P1]. The symbol $\backsim$ denotes "homotopic to" (see [P1]).

A. If α, β, γ are paths such that $\beta \backsim \gamma$ and if $\alpha\beta$ exists, then $\alpha\gamma$ exists and $\alpha\beta \backsim \alpha\gamma$. (Theorem 4.5 of [P1], with $\gamma = \delta$)

B. If α is any path, then $\alpha\alpha^{-1}$ and $\alpha^{-1}\alpha$ are homotopic to null paths. (Theorem 4.9 of [P1].)

C. If α is any path, and β is a null path such that $\alpha\beta$ exists, then $\alpha\beta \backsim \alpha$. (Theorem 4.7 of [P1])

Assertion in Case 1

If in P, avba and awba are non-contractible circuits, then the circuit avbwa is contractible.

Proof

Let the paths α, β, γ be the chains avb, ba, bwa respectively, as shown in Figure 10.8. Then the circuit avbwa is $\alpha\beta$.

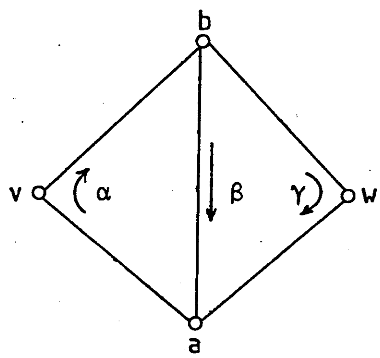


Figure 10.8

We consider $\Pi_1(P,a)$, the fundamental group of P at a, and recall that $\Pi_1(P,a)$ has two elements e, x such that $x^2 = e$, the identity element.

Now, in $\Pi_1(P,a)$, $[\alpha\beta] = x = [\beta^{-1}\gamma]$, since the closed paths $\alpha\beta$ and $\beta^{-1}\gamma$ are not contractible, that is, they are not homotopic to a null path in P (here homotopy is always taken relative to the point a).

Therefore $[\alpha\beta][\beta^{-1}\gamma] = x^2 = e$, that is $[\alpha\beta\beta^{-1}\gamma] = e$.

Now, $\beta\beta^{-1} \backsim \alpha_0$, where α_0 is the null path at a (by B above), so that $\alpha\beta\beta^{-1} \backsim \alpha\alpha_0$, by A.

Hence $\alpha\beta\beta^{-1} \backsim \alpha$, by C (and since $\backsim$ is an equivalence relation).

We deduce that $\alpha\beta\beta^{-1}\gamma \backsim \alpha\gamma$ by A, so that $[\alpha\beta\beta^{-1}\gamma] = [\alpha\gamma] = e$.

It follows that $\alpha\gamma$ is contractible in P, as required. $\square$

Assertion in Case 2

If avba, awca, bucb are non-contractible circuits in P, then either bvawcub or bvawcb is contractible.

Proof

Let the paths α, β, γ, δ, ε, ζ be as shown in Figure 10.9, and let us consider $\Pi_1(P,b)$.

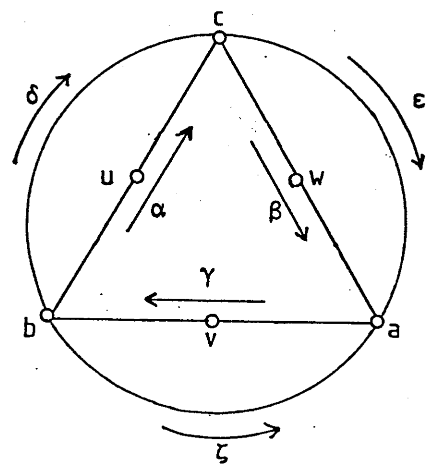


Figure 10.9

If we suppose that bvawcub is non-contractible, that is, if in $\Pi_1(P,b)$, $[\alpha\beta\gamma] = x$, then $[\delta\alpha^{-1}][\alpha\beta\gamma] = x^2 = e$, since $\delta\alpha^{-1}$ is non-contractible. Hence $[\delta\alpha^{-1}\alpha\beta\gamma] = e$. But as above, $\delta\alpha^{-1}\alpha\beta\gamma \backsim \delta\beta\gamma$, so that $[\delta\alpha^{-1}\alpha\beta\gamma] = [\delta\beta\gamma] = e$. It follows that $\delta\beta\gamma$ is contractible in P, as required. □

APPENDIX CUTVERTEX-RECONSTRUCTION OF TREES

During the Problem Session of the Seventh British Combinatorial Conference held in Cambridge in 1979, Harary conjectured that a tree is reconstructible from its subgraphs obtained by deleting one cutvertex at a time. The aim of this Appendix is to show that this conjecture is true. Although this is a variant of the Vertex-reconstruction Problem, in Section A.2 we employ the same technique involving associates which we used in Theorem 3.3 and in Chapters 7 and 10 where we were considering edge-reconstruction.

One of the first results in graph reconstruction was the proof by Kelly [Kl] that trees are vertex-reconstructible. This result was subsequently proved under more stringent conditions by various authors [Bl, HPl, M2]. In these cases only those graphs in the vertex-deck resulting from the deletion of 1-vertices were used. During the Problem Session of the Seventh British Combinatorial Conference held in Cambridge in 1979, Harary conjectured that the remaining graphs in the vertex-deck also suffice to reconstruct a tree. The aim of this appendix is to show that this conjecture is true.

We recall that a cutvertex of a graph T is a vertex whose removal increases the number of components of T. We say that T is <u>cutvertex-reconstructible</u> (<u>cv-reconstructible</u>) if it is uniquely determined, up to isomorphism, from the <u>cutvertex-deck</u>

$$\mathcal{CD}T := \{\{T_v : \ v \ \text{cutvertex of} \ T\}\}.$$

The definition of a <u>cutvertex-recognizable</u> class of graphs is analogous to that of a vertex-recognizable class.

MAIN THEOREM

Every tree with at least three cutvertices is cv-reconstructible.

The condition that the tree must have at least three cutvertices is essential as can be seen from the graphs in Figures A.1 and A.2; the two graphs in Figure A.1 have the same cutvertex-deck as do the three graphs in Figure A.2.

REMARK. If however, we were to be given, apart from $\mathcal{CD}T$, the extra information that T is a tree, then the trees in Figure A.1 and A.2 would in fact be cv-reconstructible. In fact, it is easily seen that with this extra information, trees with one or two cutvertices would be cv-reconstructible, and so in this case the restriction on the number of cutvertices could be dropped from the Main Theorem. (This theorem would then read: Trees with at least three vertices are weakly cv-reconstructible.)

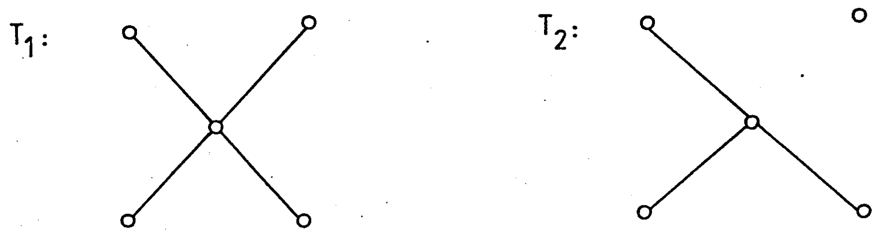


Figure A.1

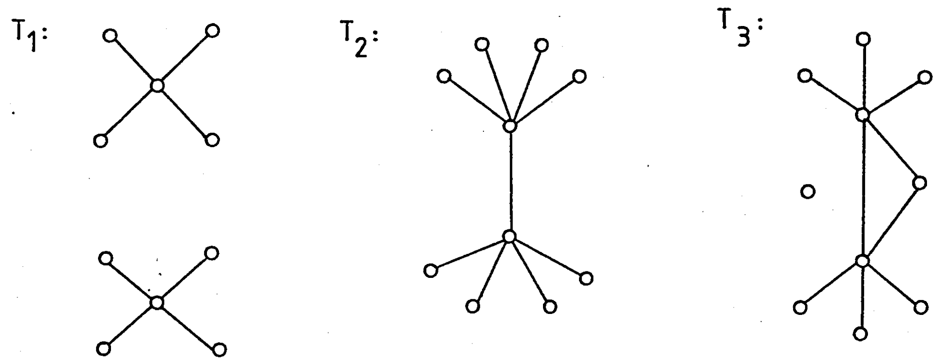


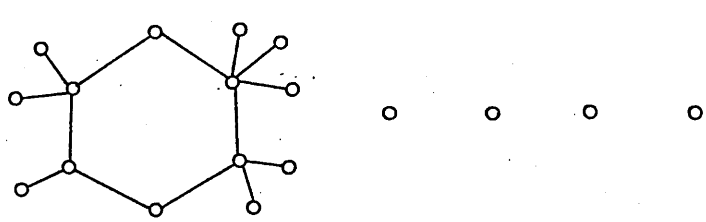
Figure A.2

SECTION A.1 - RECOGNITION

The first problem is to show that we can actually recognize from $\mathcal{CD}T$ whether or not T is a tree. We first give a few definitions. We recall that a forest is a graph in which every pair of vertices is joined by at most one chain, and that a tree is a connected forest. A 1-vertex will also be called an endvertex. An end-cutvertex of a tree is a cutvertex all of whose neighbours except one are endvertices. If F is a forest, then a nontrivial component of F is a component which has at least one cutvertex; a trivial component of F is one which has no cutvertices. For any graph T, the number of cutvertices of T is denoted by $\nu_c T$. Clearly, both νT and $\nu_c T$ are reconstructible from the cutvertex-deck of T. We now proceed to show that we can determine from $\mathcal{CD}$T whether or not T is a tree.

We observe first that if T is disconnected and has a component K such that K has more than one vertex but does not have a cutvertex, then we can easily determine from $\mathcal{CD}$T that T is not a tree. This follows because the component K appears in every $T_v \in \mathcal{CD}T$, whereas we know that if T is a tree with at least three cutvertices, then $\mathcal{CD}$T must contain at least two forests T_v, T_w (corresponding to v, w end-cutvertices) such that T_v and T_w have exactly one nontrivial component, all the other components being isolated vertices. We may therefore assume that any component of T with no cutvertices is simply an isolated vertex.

Now, it is clear that if some graph in $\mathcal{CD}$T contains a circuit, then T is not a tree. We may therefore assume that every graph in the cutvertex-deck of T is a forest. Hence, if T is not a forest (that is it contains a circuit C), then every cutvertex of T must be a vertex of C. Therefore all cutvertices of T must be in one component H of T (so that by the comments in the previous paragraph we may assume that any component of T not containing C is an isolated vertex), and this component H must be a unicyclic graph such that each block of H different from C is the trivial tree on two vertices (see [H2] for definitions of the terms "unicyclic" and "block"). An example of such a graph is shown in Figure A.3. We shall call such graphs <u>circuit-critical</u>.



<u>Figure A.3</u>

Lemma A.1

Let T be a graph with at least three cutvertices. Then T is a circuit-critical graph if and only if each T_v in $\mathcal{CD}T$ is a forest with exactly one nontrivial component, and such that every other component of T_v is an isolated vertex.

Proof

The necessity is clear. To prove the converse, let every T_v in $\mathcal{CD}T$ satisfy the conditions of the lemma, and let us suppose that T is not circuit-critical. Then, by the comments preceding the lemma, T is a forest. If T has a component with at least three cutvertices, then $\mathcal{CD}T$ contains a graph T_v, such that T_v either has more than one nontrivial component, or else contains a trivial component which is not an isolated vertex. We may therefore assume that every nontrivial component of T contains at most two cutvertices. But since T has at least three cutvertices, then it must have at least two nontrivial components (and at least three such components if each nontrivial component of T has only one cutvertex), and we again deduce that there is a graph in $\mathcal{CD}T$ which either has more than one nontrivial component or else contains a trivial component which is not an isolated vertex, a contradiction. □

We can therefore recognize from $\mathcal{CD}T$ whether or not T is a forest. Hence we are just left with the problem of determining from the cutvertex-deck whether T is a tree or a disconnected forest. We recall that if T is a forest, we may then assume that any trivial component of T is an isolated vertex. Also, if T has more than one nontrivial component, then (since T has at least three cutvertices) it is easily determined from $\mathcal{CD}T$ that T is not a tree, because $\mathcal{CD}T$ would contain at most one graph T_v having only one nontrivial component and such that all the other components of T_v are isolated vertices, whereas we know that if T is a tree, then $\mathcal{CD}T$ must contain at least two such forests (corresponding to deletions of end-cutvertices). We may therefore assume that if T is a disconnected forest, then T has exactly one nontrivial component, all the other components being isolated vertices.

Therefore we only have to determine whether or not T has any isolated vertices. If some T_v in $\mathcal{CD}T$ has no isolated vertices, then neither does T. We may therefore assume that for every T_v in $\mathcal{CD}T$, $\nu_0(T_v) > 0$.

We choose $T - v_0$ in $\mathcal{CD}T$ such that it has exactly one nontrivial component H, the other components being isolated vertices. Let $p = \sum \nu_0(T_w)$, where the summation is taken over all cutvertices w of T, $w \neq v_0$. Now, if $\nu_c H = \nu_c T - 2$, then v_0 is adjacent in T to an endvertex u of H. Hence, T would not be a tree, because if T were a tree then $\nu_0(T_u) = 0$. We may therefore assume that $\nu_c H = \nu_c T - 1$. Thus, v_0 is adjacent in T to a cutvertex of H, and so, if q is the number of endvertices of T not adjacent to v_0 in T, we deduce that $q = \nu_1 H$. It is then clear that $\nu_0 T = 0$ if and only if $p = q$. This completes the proof that for any graph T with at least three cutvertices we can recognize from $\mathcal{CD}T$ whether or not T is a tree. We state this as a theorem.

<u>Theorem A.1</u>

Trees with at least three cutvertices are cutvertex-recognizable. □

Thus, we may henceforth assume that T is a tree with at least three cutvertices.

Before proceeding with the proof of the Main Theorem we have some more definitions. A cutvertex of T will be called <u>heavy</u> if it is adjacent to at least three other cutvertices. The <u>distance</u> d(u,v) between two vertices u and v in T is the length of a shortest chain joining them. The diameter dT of T is the length of a longest chain of T. The centre of T is the set of all central vertices of T (see [H2]). We shall use the well-known result that a tree is either central or or bicentral. If T is bicentral, then the edge adjacent to both central vertices is called the central edge. A radial vertex of a tree is one which is at a maximum distance from the centre.

Given a graph T_v in $\mathcal{CD}T$, the valency of v in T is equal to the number of components of T_v. We therefore know the valencies of all the vertices of T, and hence, given T_v, we can also determine the neighbourhood valency list of v in T.

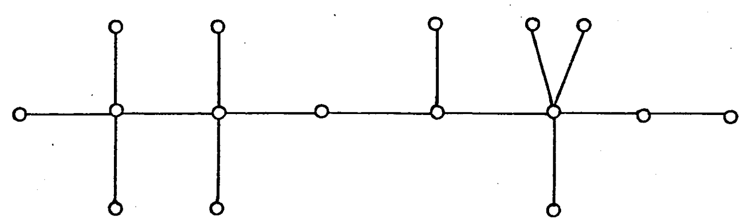
For any T_v in $\mathcal{CD}T$, dT_v denotes the maximum of the diameters of the components of T_v. If T_v has only one component with maximum diameter, then the centre of T_v is the centre of that component.

A radial cutvertex is an end-cutvertex adjacent to at least one radial vertex. A cutvertex v is called an essential cutvertex if $dT_v < dT$; otherwise it is called a non-essential (n.e.) cutvertex. We note that if an essential cutvertex is an end-cutvertex, then it is a radial cutvertex. In general, a tree can have at most two essential end-cutvertices.

A branch at a vertex v of a tree T is a maximal subtree containing v as an endvertex, and rooted at v. The central branches of a central tree are the branches of its central vertex, and the central branches of a bicentral tree are those branches of either of its central vertices which do not contain the central edge. A radial branch is one containing a radial vertex. When we use the term branch we generally mean a central branch, unless otherwise specified. If T is bicentral, then the two components (rooted at the central vertices) which result when the central edge is removed, will be called halves of T. If T_v has only one component T' with diameter equal to

dT_v, then by a branch (or radial branch or half of T_v we mean a branch (or radial branch or half) of T'.

A <u>caterpillar</u> is a tree such that the removal of all its endvertices results in a chain. Clearly, if every cutvertex of T is essential, then T is a caterpillar. Caterpillars were first studied in [HS1].



<u>Figure A.4</u> An example of a caterpillar.

In [HS1] it is shown that T is a caterpillar if and only if it does not contain $S(K_{1,3})$ as a subgraph, where $S(K_{1,3})$ is the graph obtained from $K_{1,3}$ by subdividing once each edge of $K_{1,3}$. Hence T is a caterpillar if and only if it has only two end-cutvertices, and since we can determine from CDT the number of end-cutvertices, we deduce that we can recognize from T whether or not T is a caterpillar. If T is not a caterpillar, then we can choose a T_v in CDT such that dT_v is maximal. Then $dT = dT_v$. We can therefore determine whether T is central or bicentral, and also, given any T_v, we can determine whether v is a n.e. cutvertex; T_v will be called non-essential (essential) if v is non-essential (essential). We note that for a n.e. T_v, the centre of T_v is the same as the centre of T.

We shall now proceed to prove that T is cv-reconstructible. This is carried out in three sections, according as T is a caterpillar, T is not a caterpillar and is bicentral, and T is not a caterpillar and is central.

SECTION A.2 - CATERPILLARS

In this section we shall assume that T is a caterpillar. Let the cutvertices of T be $v_1, v_2, \ldots, v_p$ such that v_1 and v_p are the end-cutvertices and v_j is adjacent to v_{j-1} and v_{j+1} for $j = 2, 3, \ldots, p-1$. We note that T is uniquely determined by the ordered p-tuple $(\rho_T v_1, \rho_T v_2, \ldots, \rho_T v_p)$ or $(\rho_T v_p, \rho_T v_{p-1}, \ldots, \rho_T v_1)$ which we shall call the <u>vector</u> of T. For example the tree in Figure A.4 is uniquely determined from either $(4,4,2,3,5,2)$ or $(2,5,3,2,4,4)$. (In [Z1] the vector of T is defined as

$$(\rho_T v_1 - 1, \rho_T v_2 - 2, \ldots, \rho_T v_{p-1} - 2, \rho_T v_p - 1).)$$

We observe that since we can determine from $T - v_i$ the valency of v_i, the valencies of the neighbours of v_i in T, and whether or not v_i is an end-cutvertex, we then know $\rho_T v_1$, $\rho_T v_2$, $\rho_T v_{p-1}$, $\rho_T v_p$. Let $a = \rho_T v_1$ and $b = \rho_T v_p$.

<u>Case 2.1</u> <u>$\rho_T v_2$ and $\rho_T v_{p-1}$ are both at least 3</u>

Let us assume that T is not cv-reconstructible and let T' be another cv-reconstruction of T, $T \neq T'$ (that is, $CDT = CDT'$). We note that T' must be a caterpillar and that its end-cutvertices must have valencies a and b.

Now, we can reconstruct from $T - v_1$ as T or as $T_1 := T - v_1v_2 + v_1v_p$ since we know that the graph to be reconstructed is a caterpillar, and since we know the valencies of the neighbours of v_1 in T. Therefore T' is the only cv-reconstruction of T, apart from T itself, and $T' \simeq T_1$.

We are thus faced with the same situation as in Theorem 3.3 and Lemmas 7.3, 7.4 and 10.2, namely that if T is not cv-reconstructible, then it has only one other cv-reconstruction apart from itself. In fact, although in the case under consideration we are not dealing with the edge-deck of T, we can still say that T and T' are associates with respect to $\{T - v_1v_2, v_1v_2, v_1v_p\}$. The only difference is that here the graph $T - v_1v_2$ is not a graph which is present in the deck under consideration. We can therefore use the same technique as we did in Theorem 3.3 and Lemmas 7.3, 7.4 and 10.2, that is, we find successive subgraphs and pairs of edges with respect to which T and T' are associates until we obtain the usual contradiction that $T \simeq T'$.

We first note that since $T_1 \simeq T'$, and since the valencies of the end-cutvertices of T' must be a and b, then the valency of v_2 in T_1 is b, therefore the valency of v_2 in T is $b+1$. We now repeat the same argument on T_1. From $T_1 - v_2$ we can reconstruct as T_1 or as $T_2 := T_1 - v_2v_3 + v_2v_1$. Therefore T and T' are associates with respect to $\{T_1 - v_2v_3, v_2v_3, v_2v_1\}$, that is, $T \simeq T_2$, so that the valency of v_3 in T_2 is a, giving that the valency of v_3 in T is $a + 1$. Continuing in this way we obtain that for $j \geq 1$, $\rho_T v_{2j} = b + 1$ and $\rho_T(v_{2j+1}) = a + 1$.

Similarly, by carrying out the same process, this time starting from $T - v_p$ instead of $T - v_1$, we deduce that for $j \geq 1$, $\rho_T(v_{p-2j}) = b+1$ and $\rho_T(v_{p-2j+1}) = a + 1$.

Hence, if p is odd, then $a = b$, and T has vector $(a,a+1,\ldots,a+1,a)$, whereas if p is even, then T has vector $(a,b+1,a+1,b+1,\ldots,a+1,b)$, where a might or might not be equal to b. But then in either case we obtain that $T \simeq T_1$, that is, $T \simeq T'$, a contradiction.

<u>Case 2.2</u> $\quad$ <u>$\rho_T v_2 = \rho_T v_{p-1} = 2$</u>

Again let us assume that T is not cv-reconstructible, and let T' be another cv-reconstruction of T, $T \neq T'$. We argue as above. This time we obtain that if T_1 is obtained from $T - v_1$ by joining v_1 to an endvertex adjacent to v_p, and to the isolated vertices of $T - v_1$, then $T_1 \simeq T'$. Therefore the valency in T_1 of v_3 is equal to b while the valency of v_4 is 2, since in T' the end-cutvertices have valencies a and b, and both are adjacent to 2-vertices. Therefore $\rho_T v_3 = b$ and $\rho_T v_4 = 2$.

Continuing in this way we obtain (in a manner similar to Case 2.1) that for $j \geq 2$, $\rho_T(v_{2j-2}) = 2$, $\rho_T(v_{4j-3}) = a$ and $\rho_T(v_{4j-5}) = b$. Similarly, if we start the above argument from $T - v_p$ instead of $T - v_1$, we obtain that for $j \geq 1$, $\rho_T(v_{p-2j+1}) = 2$, $\rho_T(v_{p-4j+2}) = a$ and $\rho_T(v_{p-4j}) = b$.

This implies that if $p = 4q + 3$, for some integer q, then T has vector $(a,2,b,2,a,\ldots,2,a,2,b)$, if $p = 4q + 1$, then $a = b$, and T has vector $(a,2,a,\ldots,a,2,a)$, otherwise $a = b = 2$ and T has vector $(2,2,2,\ldots,2,2)$. But in any case, $T \simeq T_1$, that is, $T \simeq T'$, a contradiction.

<u>Case 2.3</u> <u>$\rho_T v_2 = 2$ and $\rho_T v_{p-1} \geq 3$</u>

We argue as above, assuming that T is not cv-reconstructible and that T' is another cv-reconstruction of T, $T \not\simeq T'$. Starting the usual argument from $T - v_1$, we obtain that for $j \geq 1$, $\rho_T(v_{3j-1}) = 2$, $\rho_T v_{3j} = b$ and $\rho_T(v_{3j+1}) = a + 1$. Similarly, using the same argument this time starting from $T - v_p$, we obtain that for $j \geq 1$, $\rho_T(v_{p-3j+2}) = a + 1$, $\rho_T(v_{p-3j+1}) = 2$ and $\rho_T(v_{p-3j}) = b$.

This implies that if $p = 3q$ for some integer q, then $a + 1 = 2$, that is $a = 1$, which is impossible. If $p = 3q + 1$, then $b = a + 1$, and T has vector $(a,2,a+1,a+1,2,a+1,a+1,\ldots,2,a+1,a+1)$. If $p = 3q + 2$, then $b = 2$, and T has vector $(a,2,2,a+1,2,2,a+1,\ldots,2,2,a+1,2)$.

But if T_1 is the graph obtained from $T - v_1$ by joining v_1 to an endvertex adjacent to v_p and to the isolated vertices of $T - v_1$, we obtain that $T' \simeq T_1$. However, since T has one of the above two vectors, it then follows that $T \simeq T_1$; therefore $T \simeq T'$, a contradiction.

SECTION A.3 - BICENTRAL TREES

Throughout this section we shall assume that T is not a caterpillar and that it is bicentral.

Case 3.1 T has only one n.e. cutvertex

Let $T - v_1$ be non-essential. Since v_1 is the only n.e. cutvertex, it is an end-cutvertex. Also, if t is the other cutvertex adjacent to v_1 in T, then t is the only heavy vertex of T, and the only nontrivial component of $T - v_1$ is a caterpillar. We observe also that there are only two other T_v, $T_w \in \mathcal{CD}T$ with v and w end-cutvertices. Also, since v_1 is the only n.e. end-cutvertex, then $d(t,v) \geq 2$ and $d(t,w) \geq 2$ in T (see Figure A.5).

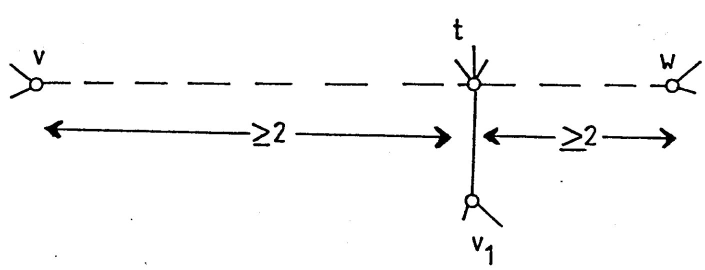


Figure A.5

Let us assume first that t is heavy in both T_w and T_v. If in both T_w and T_v, t is adjacent to only one end-cutvertex, we then choose that one of T_w and T_v in which the distance from t to the nearest non-adjacent end-cutvertex is a minimum. We may assume that this is T_w. Let x be the end-cutvertex of T_w nearest to t and not adjacent to t. We then reconstruct T from T_w by joining w to the isolated vertices of T_w and to x, if w has a neighbour with valency greater than 2, or to an endvertex adjacent to x if w has a 2-vertex as neighbour. (We recall that the valencies of the neighbours of w in T are known.)

If in one of T_v or T_w, say in T_w, t is adjacent to two end-cutvertices x', y' say, then v_1 is one of x', y' (since v_1 is the only n.e. cutvertex of T). If in T_w, x' and y' have have different valencies, then knowing $\rho_T v_1$ we can distinguish which one of x', y' is v_1, (say y' is), and we can then continue as above with x' now taking the place of x. If x' and y' have the same valencies in T_w, then we can also continue as above, choosing either one of x', y' as v_1, in either case giving isomorphic reconstructions.

We may therefore assume that T_w, say, has no heavy vertex. Hence, we know that $d(t,w) = 2$, with both t and w having a 2-vertex as a common neighbour in T (see Figure A.6).

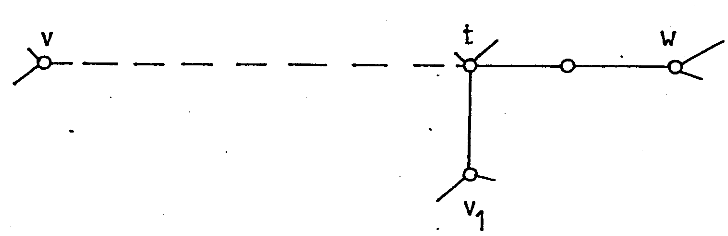


Figure A.6

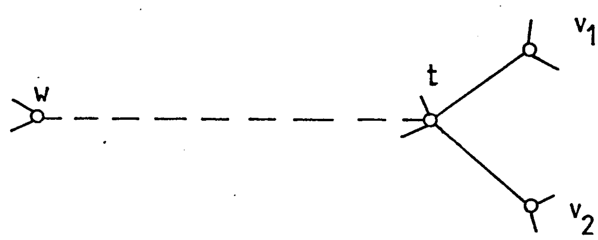
Therefore, in order that there may be any ambiguity in reconstructing T from $T - v_1$, the vertex v must be adjacent to a 2-vertex y. But then we can reconstruct uniquely from T_w, for the following two reasons: (a) we know that w is adjacent to the isolated vertices of T_w and to an endvertex which is at a distance 2 from an end-cutvertex, and (b) in T_w, the vertex t is the only cutvertex which has such an endvertex as neighbour, since y is a 2-vertex.

<u>Case 3.2</u> <u>T has only two n.e. cutvertices</u>.

Let $T - v_1$ and $T - v_2$ be non-essential.

<u>3.2.1</u> Let us assume first that both v_1 and v_2 are end-cutvertices. Therefore they are not adjacent in T.

<u>3.2.1a</u> We assume first that $T - v_1$, say, is a caterpillar. Then, since T has two n.e. cutvertices, v_1 must be adjacent to one of the two cutvertices of $T - v_1$ which are neighbours of the end-cutvertices of $T - v_1$. Therefore $T - v_2$ is also a caterpillar, and $d(v_1,v_2) = 2$, that is, v_1 and v_2 have a common neighbour t which is heavy in T. We observe also that t is the only heavy vertex of T, that $\rho_T t$ is known and that there exists only one other T_w, w end-cutvertex, apart from $T - v_1$ and $T - v_2$ (see Figure A.7).



<u>Figure A.7</u>

Hence, if $dT \geq 7$, then T is reconstructible from $\dddot{T}_w$. Since for bicentral trees the diameter is necessarily odd, the only other case we need consider is $dT = 5$. In this case we consider $T - v_1$. We know that t is one of the two central vertices of $T - v_1$, and since we know the valency of t in T, we can then assume that both central vertices have the same valency in $T - v_1$. Also, since we know the valency of v_2 in T, we can then assume that the two end-cutvertices both have the same valency $\rho_T v_2$ in $T - v_1$ (otherwise there is no ambiguity in deciding which of the two central vertices is t). But now, we can take t to be any one of the central vertices, since we obtain isomorphic reconstructions in either case.

<u>3.2.1b</u> We may therefore assume that no $T - v_i$, $i = 1, 2$, is a caterpillar. Each $T - v_i$ therefore has a unique heavy vertex t_i. We note that if $\{i,j\} = \{1,2\}$, then t_i is the neighbour of v_j in T. In fact, v_j is the n.e. cutvertex of $T - v_i$.

Let x_i be the distance between t_i and the nearest radial vertex in $T - v_i$. Now, if we can determine whether or not t_1 and t_2 are in the same half of T, we could reconstruct from either $T - v_1$ or $T - v_2$, since we know x_1 and x_2. So we now proceed to determine whether or not t_1 and t_2 are in the same half of T.

First we note that if $x_1 = x_2$, then t_1 and t_2 are in the same half of T (that is $t_1 = t_2$) if and only if T has a vertex adjacent to four cutvertices (something which we can determine). We may therefore assume that $x_1 > x_2$.

Now, if t_1 and t_2 are in the same half, then $d(t_1,t_2) = x_1 - x_2$, whereas if they are not, then $d(t_1,t_2) = dT - x_1 - x_2$. Since dT is odd, then $x_1 - x_2 \neq dT - x_1 - x_2$. Hence if we can determine $d(t_1,t_2)$ we would know whether or not t_1 and t_2 are in the same half of T.

Let T_v and T_w be the other two forests in *CD*T with v, w end-cutvertices. If at least one of them has two heavy vertices, then we can determine $d(t_1,t_2)$. But this is always so because if we assume that each of T_v, T_w has only one heavy vertex, then v is adjacent to a 2-vertex which is a neighbour of t_i, and w is adjacent to a 2-vertex which is a neighbour of t_j (see Figure A.8), and this is impossible, since then we would have that $x_1 = 2 = x_2$.

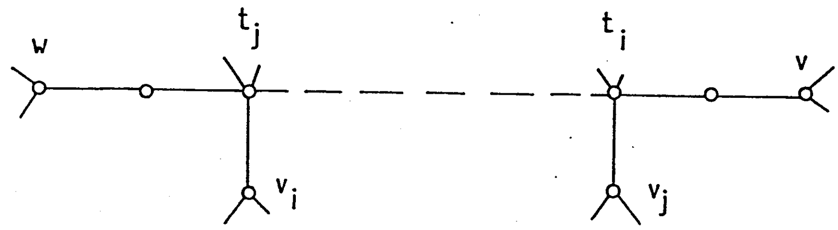


Figure A.8

3.2.2 We now assume that not both v_1 and v_2 are end-cutvertices. Therefore they are adjacent and one of them, say v_1, is an end-cutvertex. Then, if $\rho_T v_2 > 2$, we can easily identify v_2 in $T - v_1$, and hence reconstruct T. We therefore assume that $\rho_T v_2 = 2$.

Now, T has a unique heavy vertex t, which is a neighbour of v_2. Also, $\mathcal{CD}T$ has two other forests T_v, T_w with v, w end-cutvertices. Moreover, since v_1 and v_2 are the only n.e. cutvertices, then $d(t,v) \geq 3$ and $d(t,w) \geq 3$ in T.

Therefore t is heavy in both T_v and T_w, and we can proceed to reconstruct in a manner similar to Case 3.1. (That is, if in both T_v and T_w, t is adjacent to only one 2-vertex q which is a neighbour of an end-cutvertex, then we choose that graph from $\{T_v,T_w\}$ in which the chain, not passing through q, from t to the nearest end-cutvertex is the shortest, and we proceed as in Case 3.1. If in T_v, say, t is adjacent to two 2-vertices which are neighbours of end-cutvertices x' and y', then one of x', y' is v_1, and we again proceed to distinguish which one is v_1 as we did in Case 3.1 (using $\rho_T v_1$ which we know).)

Case 3.3 T has at least three n.e. cutvertices

We shall first determine whether or not all the n.e. cutvertices are in the same half of T. Towards this end we define the following nine types of tree (shown in Figure A.9).

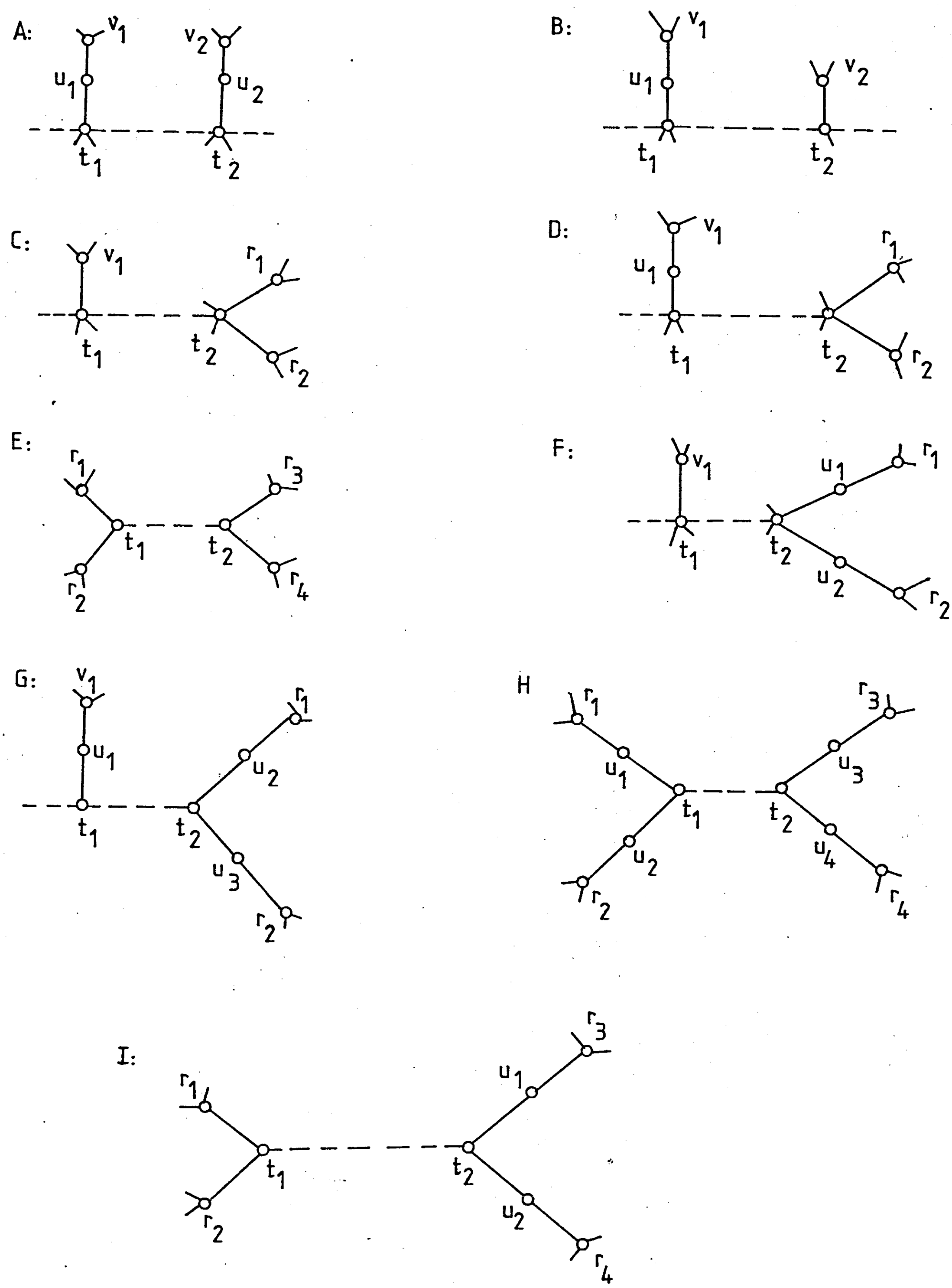


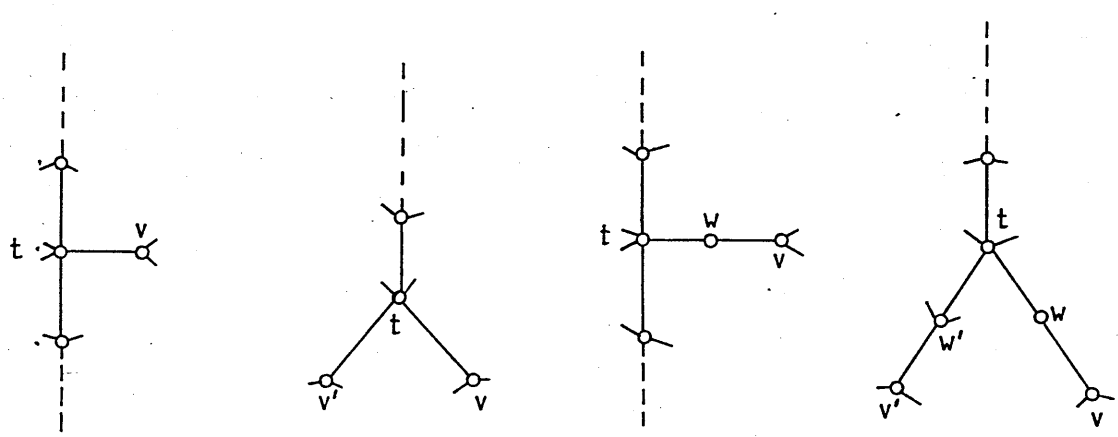
Figure A.9

In each of these types, there are exactly two heavy vertices t_1 and t_2. The only n.e. cutvertices are those labelled u_i, v_i and r_i. Those labelled u_i are 2-vertices and are not end-cutvertices, those labelled v_i are end-cutvertices but are not radial, whereas those labelled r_i are radial.

We shall see later that it is easy to recognize when T is one of the nine types of Figure A.9, and to reconstruct it in such a case. We shall therefore assume for the time being that T is not one of these nine types of tree.

Lemma A.2

Let G be a bicentral tree, and let H_1 and H_2 be the halves of G. Let v be a n.e. end-cutvertex of G, $v \in VH_1$. Then G_v contains no other n.e. cutvertex in $VH_1 - \{v\}$ if and only if v is as in one of the configurations of Figure A.10, where t is the only heavy vertex of G in H_1, w is a 2-vertex, and in Figure A.10(ii) and Figure A.10(iv), v and v' are radial.



(i) (ii) (iii) (iv)

Figure A.10

Proof

If v is in one of the above configurations, then clearly G_v has no n.e. cutvertex in $VH_1 - \{v\}$ For the converse, let us assume that G_v has no n.e. cutvertex in $VH_1 - \{v\}$. We note that VH_1 must contain a heavy vertex of G (since it contains the n.e. cutvertex v) and that $VH_1-\{v\}$ does not contain a heavy vertex of G_v. Therefore VH_1 contains only one heavy vertex t of G. We note also that t cannot be adjacent to more than three cutvertices in G, because otherwise it would still be heavy in G_v. Therefore t is adjacent to exactly three cutvertices in G.

If v is adjacent to t, then it is in configuration (i) or (ii) of Figure A.10. Therefore we may assume that v is not adjacent to t. Then since t is not heavy in G_v, v is adjacent to a 2-vertex w which is a neighbour of t. But then v is in configuration (iii) or (iv) of Figure A.10. □

Lemma A.3

Let T be not one of the nine types of tree shown in Figure A.9. Then all the n.e. cutvertices of T are in one half if and only if either in each T_v, v a n.e. end-cutvertex, all the n.e. cutvertices are in one half, or else T_v contains no n.e. cutvertices, for some v n.e. end-cutvertices.

Proof

The necessity is obvious. To prove the converse we note first that if a T_v, v a n.e. end-cutvertex, has no n.e. cutvertices, then all the n.e. cutvertices of T are in one half; similarly if T has only one subforest T_v with v a n.e. end-cutvertex. Therefore we assume that there are at least two such T_v, and each one of them contains at least one n.e. cutvertex.

Now, let us assume that the lemma is false. Then there exists a $T - v_1$, v_1 a n.e. cutvertex, with two halves H_1', H_2, where H_2 contains all the n.e. cutvertices of $T - v_1$, and such that v_1 is adjacent to a vertex of H_1' in T. Let H_1 be the half of T containing v_1, that is, $H_1' = H_1 - v$. Then in H_1, the vertex v_1 is as in one of the configurations of Figure A.10 (with $v_1 = v$). Moreover, if v_1 is as in Figure A.10(iv), then w' must be a 2-vertex, otherwise $T - v'$ would contain n.e. cutvertices in both halves.

Now, we let v_2 be a n.e. end-cutvertex in VH_2 and consider $T - v_2$. By the hypothesis of the lemma, and since v_1 is a n.e. end-cutvertex of $T - v_2$ in VH_1, then $VH_2 - \{v_2\}$ contains no n.e. cutvertex of $T-v_2$. Therefore v_2 is also as in one of the configurations of Figure A.10 in H_2 (and again, if v_2 is as in Figure A.10(iv), then w' must be a 2-vertex). But then, since we know that T must have at least three n.e. cutvertices, we deduce that T is one of the nine types of tree of Figure A.9, a contradiction. □

Now, let us consider first the case when not all the n.e. cutvertices of T are in one half. (This fact is recognizable by Lemma A.3, since we are assuming that T is not one of the nine types of tree of Figure A.9.)

We pick a $T - v_0$, v_0 a n.e. cutvertex, such that $T - v_0$ has a half which has a maximum number of n.e. cutvertices among all halves of all n.e. forests in $\mathcal{CD}T$. Then this half H_1 is a half of T.

We now pick all those T_v, v non-essential, which have no half isomorphic to H_1. (If no such T_v can be found, then both halves of T are H_1). Let these be $T - v_1, T - v_2, \ldots, T - v_s$. (We note that s, which is the number of n.e. cutvertices of T in H_1, is at least 2, since T has at least three n.e. cutvertices and H_1 has a maximal number of n.e. cutvertices.)

Let the two halves of $T - v_i$ be $H_{1,i}$, $H_{2,i}$. There should be a half which appears in each of $T - v_i$. Let this half H_2 be the one which we have called $H_{2,i}$. Therefore each $T - v_i$ has two halves $H_{1,i}$ and H_2.

Now, if there exists an $i \neq j$ such that $H_{1,i} \not\simeq H_{1,j}$, then we know that H_2 is the other half of T. Therefore we can assume that for all $i = 1, 2, \ldots, s$, $H_{1,i} \simeq K$, say. If $K \simeq H_2$, then again we conclude that H_2 is the other half of T, and so we assume that $K \not\simeq H_2$.

This situation can only arise if $H_1 - v \simeq H_1 - w$, for every v, w n.e. cutvertices of T in H_1. But we do know $H_1 - v$, for every n.e. cutvertex in H_1, since we know H_1. Let this $H_1 - v$ be H. Hence we can determine which of K or H_2 is the other half of T by choosing that one which is not isomorphic to H.

We now have to consider the case when all the n.e. cutvertices of T are in one half. We therefore have that either there exists a T_v, with v a n.e. end-cutvertex, such that in T_v one half contains at least one n.e. cutvertex, or else there does not exist such a T_v. We consider these cases separately.

<u>3.3a</u> <u>Each T_v, with v a n.e. end-cutvertex, contains no n.e. cutvertices</u>

Let us denote the two halves of T by H and H', where H' is the one which contains the n.e. cutvertices of T. Let us consider any n.e. T_v, with v an end-cutvertex. Since T_v contains no n.e. cutvertices, then v must be in one of the configurations of Figure A.10 in H', with t as the only heavy vertex of T in H'. Hence t is the only heavy vertex of T, since H does not contain any n.e. cutvertex of T. But then, since T contains at least three n.e. cutvertices, v must be as in Figure A.10(iv), and w' of Figure A.10(iv) must be a 2-vertex; otherwise $T_{v'}$ would contain w' as a n.e. cutvertex. Therefore dT is at least 7.

Now, let T_z be the essential forest in $\mathcal{CD}T$ with z an end-cutvertex. Then z is the end-cutvertex of T which is in H. If the diameter of T is at least 9, then it is easy to reconstruct from T_z. We first observe that t is heavy also in T_x. We then let $B(t)$ be the longest branch of T_z at t. If z is adjacent to a vertex with valency greater than 2 in T, we join z to the end-cutvertex of $B(t)$ not adjacent to t, whereas if z is adjacent to a 2-vertex we join it to the endvertex of $B(t)$ different from T. We can therefore assume that the diameter of T is 7. Now we note that if $\rho_T z$ is different from $\rho_T v$, then T would be reconstructible from $T_{v'}$, so that $\rho_T z = \rho_T v$. Similarly, $\rho_T z = \rho_T v'$. (We are taking v' to be the other n.e. end-cutvertex of T.) But now let us consider T_z. Again t is heavy in T_z and two of the branches of T_z at t are caterpillars with vector $(\rho_T t, 2, \rho_T v)$. If the other branch is not of this type we know that z belongs to it, and we can continue as above. If the other branch also has this vector, then we can also continue as above, putting z in either one of these three branches, since we obtain isomorphic reconstructions in any case.

3.3b There is a T_v, v a n.e. end-cutvertex, which contains at least one n.e. cutvertex

We assume first that there also exists a n.e. T_v, v an end-cutvertex, which contains no n.e. cutvertices. Then as above we conclude that v is as in Figure A.10(iv) with t the only heavy vertex of T. But then, in this case, we must have that the valency in T of the vertex w' is at least 3. It is now easy to see that T is reconstructible from $T_{v'}$. We may therefore assume that for any n.e. T_v, v end-cutvertex, there is exactly one half not containing any n.e. cutvertices of T_v. This half H is one of the two halves of T. We note that all the heavy vertices of T are in the other half, and since T has at least three n.e. cutvertices, then each heavy vertex of T is heavy in at least one n.e. T_v. We choose a n.e. T_v with a heavy vertex at a minimum distance from its centre. Let this minimum distance between heavy vertex and the centre be h. (If one of the central vertices of T is heavy, then $h = 0$.)

Let α be the end-cutvertex of T which is in H and let α' be the cutvertex which is adjacent to α. If a and b, say, are the central vertices of T, with $a \in VH$, we let H' be the graph H with the edge ab added, and rooted at b. We now consider:

(i) The valency of α' in T is greater than 2

Then T_α has diameter equal to $dT - 1$, and is therefore central. If $h = 0$, we choose a T_z, z an essential end-cutvertex, such that $dT_z = dT - 1$, and such that its central vertex is heavy. If $h > 0$, we choose T_z such that z is an essential end-cutvertex with $dT_z = dT - 1$, and such that the heavy vertex nearest to the central vertex is at a distance of $h - 1$ from it. Then in either case, T_z is T_α, and we can reconstruct from T_z by choosing a branch of T_z, isomorphic to $H' - \alpha$, and joining z to the isolated vertices and to the end-cutvertex of this branch.

(ii) The valency of α' in T is equal to 2

Then again we choose an essential T_z, z an end-cutvertex, such that the heavy vertex of T_z is as near as possible to the centre. Then this graph is T_α. We now consider two subceses.

(ii.1) $h \geq 1$

Then T_α has diameter equal to $dT - 2$, that is, it is bicentral. But then we choose the half of T_α which contains no heavy vertices, and we join α to the isolated vertices of T_α and to a radial endvertex in this half.

(ii.2) $h = 0$

Then T_α can be central (that is, $dT_\alpha = dT - 1$). This can only arise if the central vertex of T which is heavy has at least two radial branches. But in this case we choose a branch of T_α which is isomorphic to $H' - \alpha$, and we join α to the isolated vertices and to a radial endvertex of this branch. We may therefore assume that T_α is bicentral, with $dT_\alpha = dT - 2$.

Let u and u' be the two central vertices of T_α. One of them, at least, must be heavy (since $h = 0$). If only one of them is heavy, say u is, then we choose a branch at u, isomorphic to $H' - \alpha$, and join α to the isolated vertices of T_α and to a radial endvertex of this branch. We therefore assume that both u and u' are heavy.

Now, at least one of u or u' must have a branch isomorphic to $H' - \alpha$. If only one of u or u' has such a branch, then we proceed as above. Therefore we assume that both u and u' have such a branch. Now, let A be the family of branches of T_α at u, and similarly let B be the family of branches at u'. Then if $A = B$, there is no ambiguity in choosing the branch $H' - \alpha$ at either u or u' (since we would obtain isomorphic reconstructions). Then let us assume that $A \neq B$.

Let us return for a moment to some considerations on T. Let $dT = 2r+1$ and let x, y be the central vertices of T. We know that exactly one of x, y, say y, is heavy (we know that $y \in \{u,u'\}$, but we have to determine whether $y = u$ or $y = u'$). Also, in T, y can have only one radial branch, because otherwise $dT_\alpha = dT - 1$. Now, let $\{\lambda,\mu\} = \{u,u'\}$. If λ is y, then μ is in the radial branch of y $(=\lambda)$ in T. We now proceed as follows.

Let T_v be non-essential, with one of the central vertices heavy (therefore this vertex is y). Let C_v be the family of non-radial branches of T_v at y. We choose a n.e. $T - v_0$, such that the total number of all the branches of C_{v_0} gives a maximal value. Then, C_{v_0}

is the family of all non-radial branches of T at y. (This is so because we know that if $\{\lambda,\mu\} = \{u,u'\}$, and $\lambda = y$, then μ is in the radial branch of y, and since μ is heavy, then there is at least one n.e. cutvertex μ' adjacent to μ, so that $C_{\mu'}$ would certainly give us the required C_{v_0}.)

But now let $C = C_{v_0} \cup \{H' - \alpha\}$. Then either $C = A$ or $C = B$. We may assume that $C = A$, so that $y = u$. Then in T_α, we join α to a radial endvertex of a branch of u isomorphic to $H' - \alpha$.

We are now left with the task of proving reconstruction in the nine cases shown in Figure A.9. We first recall that we know the number of n.e. cutvertices of T, the number of them which are end-cutvertices, the number of heavy vertices, and for each of these, we know its valency and the valencies of its neighbours in T. It is therefore easy to see that we can determine whether or not T is one of the types of tree shown in Figure A.9, and if it is , then what type it actually is. We now briefly discuss for each case how T is reconstructed.

We note first that if T is one of types E, H, I, then it is easily reconstructible from either $T - r_1$ or $T - r_2$. We then consider the other cases.

Type A Let $T - x_1$ and $T - x_2$ be the essential forests in $\mathcal{CD}$T with x_1, x_2 end-cutvertices. We note that since v_1, v_2, u_1, u_2 are the only n.e. cutvertices of T, then $d(t_i,x_j)$, $i,j \in \{1,2\}$, is at least 3 in T. Therefore both $T - x_1$ and $T - x_2$ contain two heavy vertices, so that we can determine $d(t_1,t_2)$.

Now, given $T - x_i$, $i = 1,2$, we call its two heavy vertices $z_{1,i}$, $z_{2,i}$, and we let $\ell_{1,i}$ be the maximum distance (along a chain not containing $z_{2,i}$) between $z_{1,i}$ and the nearest end-cutvertex; similarly we define $\ell_{2,i}$. Let $\ell = \text{maximum}\{\ell_{i,j}: i = 1,2;\ j = 1,2\}$. Then we know that in

T one of the heavy vertices t_1, t_2 is at a distance ℓ from an essential end-cutvertex, along a chain not containing the other heavy vertex. But then, since dT is odd, knowing $d(t_1,t_2)$ and ℓ, it is easily seen that we can reconstruct T from either one of $T - v_1$ or $T - v_2$.

<u>Type B</u> Again let $T - x_1$ and $T - x_2$ be as for type A above. As above we have that $d(t_1,x_i) \geq 3$, for $i = 1,2$, so that at least one of $T - x_1$ or $T - x_2$ has two heavy vertices, and hence we can determine $d(t_1,t_2)$. If both $T - x_1$ and $T - x_2$ have two heavy vertices, we proceed as we did for type A. If $T - x_2$, say, has only one heavy vertex, we then know that $d(t_2,x_2) = 2$, and so, knowing $d(t_1,t_2)$, we can reconstruct from $T - v_1$.

<u>Type C</u> If we know $d(t_1,t_2)$, we can easily reconstruct from $T - v_1$. Hence let T_x be the essential forest in $\mathcal{CD}T$ with x an end-cutvertex. If in T_x we see two heavy vertices, we can determine $d(t_1,t_2)$; if not, we then know that $d(t_1,x) = 2$, and knowing dT we can find $d(t_1,t_2)$.

<u>Types D,F,G</u> are dealt with in exactly the same way as type C, except that for types D and G, we must have that $d(t_1,x) \geq 3$ (since x essential), so that T_x has two heavy vertices, and we can immediately determine $d(t_1,t_2)$.

SECTION A.4 - CENTRAL TREES

Throughout this section we shall assume that T is not a caterpillar and that it is central. We now consider various cases, keeping in mind that the central vertex of T is essential.

<u>Case 4.1</u> <u>T has only one essential cutvertex</u>

In this case we can identify T_c, c the central vertex of T, as being the only essential forest in $\mathcal{CD}T$. Moreover, T must have at least three radial branches at c. From T_c we know the valency ρc of c, and the number q of endvertices adjacent to c. Therefore T has $\rho c - q$ nontrivial (that is not isomorphic to K_2) branches. Let $\mathcal{B}$ be the family of nontrivial branches of T. We need to determine $\mathcal{B}$. To this end we note that there are exactly $\rho c - q$ cutvertices $v_1, v_2, \ldots, v_{\rho c-q}$ such that each $T - v_i$ is non-essential and such that the valency of c in each $T - v_i$ is equal to $\rho c - 1$. Let $\mathcal{B}_i$ be the family of nontrivial branches of $T - v_i$. Then $\{\{\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_{\rho c-q}\}\}$ is the family of all subfamilies of $\mathcal{B}$ which have $\rho c - q - 1$ elements (that is, nontrivial branches), and from this family, $\mathcal{B}$ can be reconstructed.

<u>Case 4.2</u> <u>T has more than one essential cutvertex</u>

This means that T has only two radial branches, because otherwise the centre of T would be the only essential cutvertex. We note also that $dT > 4$, since T has only two radial branches and it is not a caterpillar.

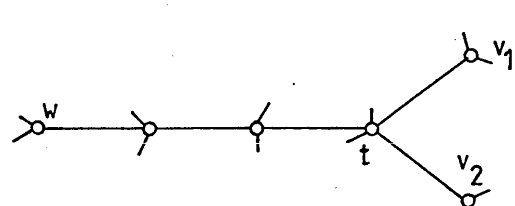
<u>4.2.1</u> <u>T has only one n.e. cutvertices</u>

This is exactly the same as Case 3.1.

<u>4.2.2</u> <u>T has only two n.e. cutvertices</u>

Again this case is almost identical to Case 3.2. We shall go over the different subcases briefly.

Case 3.2.1a carries over to when T is central, except that now, difficulty in reconstructing from T_w arises when $dT = 6$ (see Figure A.11).



<u>Figure A.11</u>

However, since we can distinguish T_w, T_t, $T - v_1$, $T - v_2$, so that we know the valencies of w, t, v_1, v_2 and the valencies of the neighbours of t and w in T, we conclude that T is uniquely determined.

The arguments in Case 3.2.1b also apply when T is central, but now, instead of determining whether or not t_1 and t_2 are in the same half of T, we determine whether or not they are in the same radial branch. However in this case t_1 or t_2 can be the central vertex of T. But if on the one hand $t_1 = t_2 = c$, the central vertex (we can obviously determine when this is so), then T is reconstructible from either $T - v_1$ or $T - v_2$. If on the other hand only t_1, say, is the central vertex, then T is reconstructible from $T - v_2$. We can therefore assume that neither t_1 not t_2 is the central vertex of T. In this case, continuing as in Case 3.2.1b, we still obtain that $dT - x_1 - x_2$ is not equal to $x_1 - x_2$, because otherwise $dT = 2x_1$, so that t_1 would be the central vertex.

Case 3.2.2 applies without modification to when T is bicentral.

<u>4.2.3</u> <u>T has at least three n.e. cutvertices</u>

Let $T - v_1$, $T - v_2, \ldots, T - v_p$ be the non-essential forests in $\mathcal{CD}T$, so that $p \geq 3$. We recall that the central vertex c can be identified in each of $T - v_i$.

<u>4.2.3.1</u> <u>The vertex c has valency k in each one of $T - v_i$</u>

Then in T, the valency of c is either k or $k+1$. But $\rho_T c = k+1$ if and only if all the n.e. cutvertices of T are adjacent to c. This is the case if and only if in T each n.e. cutvertex is an end-cutvertex adjacent to c. We can determine that this is so because this arises if and only if, in each $T - v_i$, c is the only heavy vertex and c is adjacent to $p - 1$ end-cutvertices and $(k + 1) - p - 2$ endvertices ($(k + 1) - p - 2$ could be equal to zero). But in this case

T is easily reconstructible from any $T - v_i$, $i = 1,2,\dots,p$. Thus we may assume that $\rho_T c = k$.

Therefore no n.e. cutvertex is adjacent to c, that is, c is adjacent to $k - 2$ endvertices ($k - 2$ could be equal to zero), so that c is not heavy in T. Therefore we only have to determine the two radial branches of T.

The proof of reconstruction now proceeds in a way similar to that of Case 3.3 where T was bicentral, except that now we have to determine the two radial branches of T instead of the two halves. Again we define the nine types of tree of Figure A.9, and again we see that in each case T is reconstructible in exactly the same way as in the bicentral case. (In Case 3.3, to prove reconstruction for type A of Figure A.9 we used the fact that since T was bicentral, then dT was odd. In the case under consideration, once we have determined $d(t_1,t_2)$ and ℓ as in Case 3.3, we can still say that T is reconstructible from either $T - v_1$ or $T - v_2$, this time because neither t_1 nor t_2 can be the central vertex. Similar considerations apply for type B.) Also, Lemmas A.2 and A.3 apply, where now, instead of "half of T" we have "radial branch of T".

When not all the n.e. cutvertices of T are in one radial branch, reconstruction proceeds in exactly the same way as for the corresponding bicentral case. Therefore let us assume that all the n.e. cutvertices of T are in one radial branch of T. Again we have to consider separately the two cases corresponding to Case 3.3a and 3.3b.

4.2.3.1a Each $T - v$, with v a n.e. end-cutvertex, contains no n.e. cutvertex

Let us call the two radial branches of T, H and H', where H' is the one which contains the n.e. cutvertices of T. We consider any n.e. $T - v$, v an end-cutvertex. Then we again have (as in Case 3.3a) that v must be as in Figure A.10(iv) in H', t being the only heavy vertex of T, and w' a 2-vertex in T. Then dT (which in this case is necessarly even) is at least 8, as otherwise t would be the centre of T, which is impossible since we have that the centre c is not heavy. If $dT \geq 10$, we continue as in Case 3.3a when the diameter of T was at least 9, while if $dT = 8$, we continue as in Case 3.3a when the diameter of T was 7.

4.2.3.1b There is a $T - v$, v a n.e. end-cutvertex, which contains at least one n.e. cutvertex

If there also exists a $T - v$, v a n.e. end-cutvertex, which contains no n.e. cutvertices, then we can reconstruct in the same way as was shown in the beginning of Case 3.3b. We may therefore assume that each $T - v$, with v a n.e. end-cutvertex, has exactly one radial branch not containing a n.e. cutvertex. This branch H is one of the two radial branches of T. Let α be the end-cutvertex of T which is in H, and let α' be the cutvertex which is adjacent to α. If c is the central vertex of T, let H' be the graph obtained from H by adding $k - 2$ endvertices adjacent to c, and let H'' be the graph obtained from H' by adding another endvertex adjacent to c, and this time having H'' rooted at one of the endvertices adjacent to c.

We choose a non-essential forest in $\mathcal{CD}T$ having a heavy vertex at a minimum distance from its centre, and we let this minimum distance between the heavy vertex and the centre be h, where we note that $h > 0$. We now have two cases to consider.

(i) The valency of α' in T is greater than 2

(This is exactly the same as (i) of Case 3.3b, apart from the obvious modifications.) The diameter of T_α is equal to $dT - 1$, and T_α is therefore bicentral. We choose a T_z, z essential end-cutvertex, such that the heavy vertex nearest to the centre is at a distance $h - 1$ from the centre. Then T_z is T_α, and we can reconstruct by joining α to the isolated vertices of T_α and to the end-cutvertex of that half of T_α which is isomorphic to $H' - \alpha$.

(ii) The valency of α' in T is equal to 2

If T has two essential T_z with z an end-cutvertex, we choose the one which has a heavy vertex nearest to its centre. That one is T_α. (If T has only one essential T_z, z an end-cutvertex, then it is of course T_α). We have therefore identified T_α. But we note that here, unlike (ii) of Case 3.3b, T_α always has diameter equal to $dT - 2$, since $h > 0$. Therefore T_α is central, so that we choose a branch of T_α isomorphic to $H'' - \alpha$, joining α to the isolated vertices of T_α, and to a radial endvertex of this branch.

4.2.3.2 The vertex c has valency k in some of the n.e. $T - v_i$, and valency $k - 1$ in the others

We therefore know that $\rho_T c = k$. We pick one of the $T - v_i$, $i = 1,2,\ldots,p$, in which c has valency $k - 1$. Then, from this we can determine the two radial branches of T, the number of endvertices adjacent to c (that is, the number of trivial branches of T), and all the other nontrivial, nonradial branches of T, except one, which we shall call B. If T has other nontrivial, nonradial branches apart from B, we can then reconstruct them all (as in the proof of Case 4.1) from the n.e. $T - v_i$ having the vertex c with valency $k - 1$. We may therefore assume that the only nontrivial, nonradial branch of T is B (and we can recognize this fact because we find only one n.e. $T - v_i$ with c having valency $k - 1$).

Now, if one of the radial branches has a n.e. cutvertex z, we can determine B from T_z. We do this by searching among all the $T - v_i$

for one in which the valency of c is k, and which does not have the same two radial branches as T_1 has. We then discard the radial branches and the trivial branches of this graph, and we are left with B.

We may therefore assume that both radial branches of T contain no n.e. cutvertices, and are therefore caterpillars. Let B_1 and B_2 be these two radial branches. Now, let us assume first that one of B_1 or B_2 has an end-cutvertex (apart from the one adjacent to c) which is adjacent to a vertex of valency greater than 2. We then look for some essential T_w, w an end-cutvertex, such that $dT_w = dT - 1$. Therefore T_w is bicentral. Let a and b be the two central vertices of T_w. Then, since both B_1 and B_2 are caterpillars, only one of a or b is heavy in T_w. We assume that a is heavy, so that a is c. Let L be the half of T_w not containing a, and L' the graph obtained from L by adding the edge ab and rooting it at a. Then $L' \simeq B_1$ or $L' \simeq B_2$ (no ambiguity arises in the following if $B_1 \simeq B_2$). We assume without loss of generality that $L' \simeq B_1$. Then among all the branches of T_w at a, one of them is isomorphic to $B_2 - x$ where x is the end-cutvertex of B_2 not adjacent to c. The other branches at a are the trivial branches (if any) and B.

We may therefore assume that both end-cutvertices of B_1 and B_2 not adjacent to c are adjacent to 2-vertices.

Now, let us assume that, for some end-cutvertex u, T_u has diameter $dT - 1$ (this can only arise if B has a vertex z such that $d(z,c)$ is equal to $\frac{1}{2}(dT) - 1$). In this case we can proceed as above.

We can therefore assume that for any essential end-cutvertex u, $dT_u = dT - 2$. We pick such a T_u, which is therefore central. Let y be its central vertex. Then T_u has only one heavy vertex, y' say, which is adjacent to y. Hence y' is c, the central vertex of T. Let W be the branch at y' which contains y. Then W is isomorphic to B_1 or B_2. Let us assume that it is isomorphic to B_1. Then, among the other branches of T_u at y', one of them is $B_2 - x$ (where x is the end-cutvertex of B_2 not adjacent to c), and the others consist of the branch B and all the trivial branches which T might have. So again we have that B can be determined.

This final case concludes the proof of the Main Theorem.

CONCLUDING REMARKS. When Kelly [K2] first showed that trees are vertex-reconstructible, the whole deck of vertex-deleted subgraphs was used, and it was later shown that a tree can be reconstructed from the family of its endvertex-deleted subgraphs only [HP1]. In view of this it is interesting to observe that, in most of this Appendix, only those T_v, v an end-cutvertex, were used to reconstruct T, other subforests of T being used in a few cases (notably to shorten the proofs in Section A.4 by reducing them to arguments very similar to those in Section A.3). We conjecture that in fact T can be reconstructed from the family of its end-cutvertex-deleted subgraphs, although this might make the proofs in Section 4 somewhat longer:

<u>Conjecture</u>

A tree T is reconstructible from the family

$$\{\{T_v: \ v \text{ end-cutvertex of } T\}\}.$$

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INDEX OF SYMBOLS

INDEX OF DEFINITIONS

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